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SOME TOPICS IN LIPSCHITZ ANALYSIS ON METRIC SPACES

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PRESENTADA POR

Estibalitz Durand Cartagena

Bajo la dirección del doctor

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Some topics in Lipschitz Analysis on metric spaces

Algunos aspectos del Análisis Lipschitziano en espacios métricos

Estibalitz Durand Cartagena

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Jesús Ángel Jaramillo Aguado

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Resumen

En los últimos años, el cálculo de primer orden desarrollado clásicamente en el marco de los espacios euclídeos, se ha extendido a espacios que no necesariamente están dotados de una estructura diferenciable; muchos de los avances en esta línea se pueden consultar en los trabajos de Heinonen [55, 56], Ambrosio-Tilli [4], Hajłasz-Koskela [54] o Semmes [94]. El estudio de los *espacios métricos de medida*, es decir, espacios métricos dotados de una medida de Borel, es rico en aplicaciones dentro de diferentes áreas del análisis matemático, como por ejemplo en la teoría no-lineal del potencial [10],[11],[99],[76], los grupos de Carnot [23],[84],[7], la teoría de las aplicaciones casi-conformes y casi-regulares [59],[58],[57], ciertos resultados estructurales sobre (no) inmersiones de espacios métricos [88],[26], el análisis en fractales [94],[32],[101] o el análisis en grafos [30]. Véanse también [54] y [56] y las referencias bibliográficas que aparecen en dichos trabajos.

A finales de los años setenta ya estaba claro que gran parte del análisis que involucra simplemente a las funciones (y no a sus derivadas), podía ser desarrollado en el contexto de espacios (casi-)métricos dotados de una medida de Borel *doblante* (también llamados espacios de tipo homogéneo); véanse por ejemplo [28, 29]. Una medida de Borel no trivial sobre un espacio métrico se dice *doblante* si la medida de la bola de radio r está controlada de manera uniforme por la medida de la bola de radio $2r$. Resultados fundamentales como el teorema de diferenciación de Lebesgue, el teorema del recubrimiento de Vitali o el teorema maximal de Hardy-Littlewood, válidos en en el ámbito de los espacios euclídeos, siguen siendo ciertos para espacios métricos que admiten una medida doblante.

Sin embargo, la estructura de los espacios métricos dotados de una medida doblante ha resultado ser demasiado pobre a la hora de intentar desarrollar cálculo de primer orden en dichos espacios y se hace necesario imponer otro tipo de restricciones. Por ejemplo, se puede ver que la ausencia de curvas en muchos de los fractales clásicos representa un obstáculo a la hora de dar una noción razonable de derivada. Una característica muy útil del espacio euclídeo n-dimensional, $n \geq 2$, es el hecho de que todo par de puntos x e y pueden ser unidos no sólo por el segmento $[x, y]$, sino también por una gran familia de curvas cuya longitud es comparable a la distancia entre dichos puntos. Una vez encontrada dicha familia "gruesa" de caminos, la estructura euclídea no juega ya un papel relevante y se pueden deducir de manera abstracta desigualdades fundamentales tales como la de Sobolev o la de Poincaré. Una gran parte del cálculo de primer orden que se ha desarrollado hasta ahora ha sido bajo la hipótesis de que nuestro espacio admita una *desigualdad p-Poincaré* (véase Definición 1). Dicha desigualdad crea una conexión entre la métrica, la medida y el módulo del gradiente. Además, nos proporciona un nexo de unión entre el comportamiento global y local de las funciones, es decir, podemos controlar la función en términos de su derivada.

Recordemos que dado un espacio métrico (X, d) una función $f : X \longrightarrow \mathbb{R}$ es *Lipschitz* si existe una constante C tal que

$$|f(x) - f(y)| \leq Cd(x, y),$$

para todo $x, y \in X$. En varios de los aspectos del análisis matemático en espacios euclídeos, las funciones Lipschitz han desempeñado un papel muy importante. La utilidad de este concepto resulta aún más relevante cuando no es posible hablar de "funciones diferenciables". En cierto sentido, las funciones Lipschitz son el sustituto natural de las funciones diferenciables en espacios métricos. De hecho, un resultado bien conocido de Análisis Real, el teorema de Rademacher, asegura que las funciones Lipschitz en $\mathbb{R}^n$ son diferenciables en casi todo punto del espacio con respecto a la medida de Lebesgue. Como se puede apreciar, la condición de Lipschitz es una condición puramente geométrica que cobra sentido en los espacios métricos con el simple hecho de cambiar la métrica euclidea por la métrica de nuestro espacio X y proporciona además *información global* sobre el espacio.

Uno de los resultados más sorprendentes debido a Cheeger [24] (véase también Keith [70]), es que los espacios métricos de medida dotados de una medida doblante y que admiten una desigualdad p-Poincaré, admiten una"estructura diferenciable medible" con respecto a la cual las funciones Lipschitz son diferenciables en casi todo punto. La existencia de dicha estructura es un ejemplo de restricción geométrica que tienen los espacios que admiten una desigualdad p-Poincaré. Un aspecto clave del trabajo de Cheeger es un análisis cuidadoso del comportamiento infinitesimal de las funciones Lipschitz.

Por otra parte, la noción de derivada proporciona *información infinitesimal*: se encarga de medir la oscilación infinitesimal de la función alrededor de un punto dado. Sin embargo, un espacio métrico no está dotado en general de manera natural de una estructura lineal o diferenciable y no podemos hablar por tanto de derivada, ni siquiera en el sentido débil de los espacios de Sobolev. No obstante, si f es una función con valores reales definida sobre un espacio métrico (X, d) y x es un punto de X, se pueden usar formas similares de medir las oscilaciones de primer orden de la función f alrededor de x a pequeña escala , tales como

$$D_r f(x) = \frac{1}{r} \sup \Big\{ |f(y) - f(x)| : y \in X, d(x, y) \leq r \Big\}.$$

Aunque esta cantidad no encierra tanta información como la derivada estándar en los espacios euclídeos (ya que omitimos los signos), cobra sentido en contextos mucho más generales puesto que no necesitamos ninguna condición extra sobre nuestro espacio para poder definirla. De hecho, si nos fijamos en el límite superior de dicha expresión cuando r tiende a 0, recuperamos en muchos casos, como en el contexto Euclídeo o Riemanniano, la noción estándar de módulo de la derivada. En particular, dada una función continua $f : X \to \mathbb{R}$, la *constante de Lipschitz puntual* en $x \in X$ se define de la siguiente manera:

$$\operatorname{Lip} f(x) = \limsup_{r \to 0} D_r f(x) = \limsup_{\substack{y \to x \\ y \neq x}} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Recientemente, este funcional ha jugado un papel muy importante en diferentes contextos. Mencionemos por ejemplo la construcción de estructuras diferenciables en el contexto de espacios métricos de medida [24],[70], y el teorema de diferenciabilidad de Stepanov [6]. Por otra parte, Heinonen y Koskela [59, 60] introdujeron el concepto de “gradiente superior” que juega el papel de derivada en un espacio métrico X. Una función de Borel no negativa $g : X \to [0, \infty]$ se dice que es un *gradiente superior* de $f : X \to \mathbb{R}$ si para toda curva rectificable $\gamma : [a, b] \to X$ tenemos que

$$|f(\gamma(a)) - f(\gamma(b))| \leq \int_\gamma g.$$

Observemos que el funcional $\operatorname{Lip} f$ es un gradiente superior para toda función de Lipschitz f.

Este concepto da lugar a los *espacios de funciones puntualmente Lipschitz* , que proporcionan en algún sentido información infinitesimal sobre un espacio métrico. En concreto, consideraremos los siguientes espacios de funciones:

- ⋄ $D(X) = \{f : X \longrightarrow \mathbb{R} : \|\operatorname{Lip} f\|_\infty < +\infty\}$
- ⋄ $D^\infty(X) = \{f : X \longrightarrow \mathbb{R} : \|\operatorname{Lip} f\|_\infty < +\infty \text{ y } \|f\|_\infty < \infty\}$.

El espacio $D(X)$ contiene claramente al espacio de las funciones Lipschitz $\operatorname{LIP}(X)$, y respectivamente $D^\infty(X)$ contiene al espacio de las funciones Lipschitz y acotadas $\operatorname{LIP}^\infty(X)$.

Un problema interesante es saber bajo qué condiciones se puede usar la información que se conoce infinitesimalmente para obtener información global. Uno de los objetivos de esta tesis es presentar una serie de nuevos resultados que clarifican cuándo el comportamiento infinitesimal de las funciones Lipschitz nos proporciona información sobre el comportamiento global en el contexto métrico. Este es el contenido del Capítulo II. La idea clave es requerir que los espacios sobre los que las funciones están definidas sean “altamente” conexos por caminos, lo que significa que deben existir muchos caminos que conecten cualquier par de puntos del espacio. Concretamente, en el Corolario II.1.4 daremos condiciones suficientes sobre el espacio X para garantizar la igualdad de los espacios $D(X)$ y $\operatorname{LIP}(X)$. En particular, obtendremos un resultado positivo para la clase de los espacios casi-convexos. Recordemos que un espacio es *casi-convexo* si existe una constante de tal modo que todo par de puntos se puede conectar por una curva cuya longitud no excede dicha constante multiplicada por la distancia entre los puntos. Estos espacios serán muy útiles a la hora de probar la implicación (parcial) contraria del Corolario II.1.4. Además, presentaremos una serie de ejemplos para los cuales $\operatorname{LIP}(X) \neq D(X)$ (véanse los Ejemplos II.1.5 y II.1.6).

Llegados a este punto, parece natural abordar el problema de determinar qué tipo de espacios pueden ser clasificados a partir de su estructura de Lipschitz puntual. Nuestra estrategia será seguir el enfoque desarrollado en [47], donde los autores encuentran una amplia clase de espacios métricos para los cuales el álgebra de las funciones de Lipschitz acotadas determina la estructura Lipschitz del espacio X. Un punto crucial de la prueba es el uso de la estructura de Banach del espacio $\mathrm{LIP}(X)$. Por tanto, nuestro primer cometido será dotar al espacio $D(X)$ con una norma que se define de modo natural a partir del operador Lip. En general, esta norma no es completa, como se muestra en el Ejemplo II.2.4. Sin embargo, el Teorema II.2.3 afirma que existe una clase grande de espacios, los espacios métricos *localmente radialmente casi-convexos* (ver Definición II.2.1), para los cuales $D^\infty(X)$ admite la estructura de Banach deseada. Es más, para dicha clase de espacios obtenemos un teorema de Banach-Stone en este contexto (ver Teorema II.3.6).

También estudiaremos las *isometrías puntuales* entre espacios métricos, relacionadas con las funciones de Lipschitz. En el Lema II.4.3 probaremos que dados dos espacios métricos completos localmente radialmente casi-convexos (X, d_X) y (Y, d_Y) existe un isomorfismo de retículos vectoriales entre los espacios $D^\infty(X)$ y $D^\infty(Y)$ que es una isometría para sus respectivas normas si y sólo si X e Y son puntualmente isométricos. Es claro que si dos espacios métricos son localmente isométricos, entonces son puntualmente isométricos. Sin embargo, como muestra el Ejemplo II.4.4, la implicación contraria no es cierta en general.

Además, estudiaremos en los Corolarios II.2.10 y II.2.11 el problema de caracterizar cuándo el espacio de las funciones (acotadas) puntualmente Lipschitz admite una *linealización de Banach*. Recordemos que la linealización es una herramienta útil en el estudio de los espacios de funciones, ya que nos permite aplicar el análisis funcional lineal a problemas relacionados con funciones no-lineales. Mostraremos además en el Ejemplo II.2.12 que existen espacios métricos (X, d) para los cuales $D^\infty(X)$ es un espacio dual pero no admite una linealización de Banach para X.

Si ahora tenemos una medida de Borel definida sobre nuestro espacio métrico, podemos abordar muchos más tipos de problemas. En esta línea, existen por ejemplo generalizaciones de los espacios clásicos de Sobolev en el marco de espacios métricos de medida arbitrarios. Hajłasz fué el primero en introducir los espacios de Sobolev en este contexto [52]. Hajłasz definió los espacios $M^{1,p}(X)$ para $1 \leq p \leq \infty$ en relación con los operadores maximales. Es bien conocido el hecho de que el espacio $M^{1,\infty}(X)$ coincide con el espacio de las funciones Lipschitz acotadas sobre X. Shanmugalingam introdujo en [96], usando la noción de gradiente superior (y más generalmente el concepto de gradiente superior débil), los *espacios de Newton-Sobolev* $N^{1,p}(X)$ para $1 \leq p < \infty$. Para más información acerca de los diferentes tipos de espacios de Sobolev sobre espacios métricos de medida se puede consultar [53]. En el Capítulo III de la tesis, extendemos la definición de *espacio Newtoniano* para el caso $p = \infty$ y estudiaremos las principales propiedades del espacio $N^{1,\infty}(X)$. Antes de definir los espacios $N^{1,\infty}(X)$ necesitaremos los conceptos y principales propiedades correspondientes de ∞-*módulo de una familia de curvas* (ver Definición III.2.1) y del ∞-*gradiente superior débil* (ver Definición III.2.7). Además definiremos la

∞-*capacidad* (ver Definición III.2.13), un ingrediente muy útil a la hora de probar que $N^{1,\infty}(X)$ es un espacio de Banach. Este hecho se probará en el Teorema III.2.17.

Uno de los resultados principales del Capítulo III es el Teorema III.3.3. Dicho teorema nos permite construir, en los espacios métricos dotados de una medida doblante y que admiten una desigualdad p-Poincaré, curvas casi-convexas que "evitan" conjuntos de medida cero. Este es un resultado técnico que nos permite comparar los espacios $D^\infty(X)$ y $\mathrm{LIP}^\infty(X)$ con el espacio de Sobolev $N^{1,\infty}(X)$. En particular, para la clase de espacios métricos dotados de una medida doblante y que admiten una desigualdad p-Poincaré probaremos en el Corolario III.3.5 la igualdad de todos los espacios arriba mencionados. Además, si sólo requerimos una desigualdad p-Poincaré local y uniforme obtendremos en el Corolario III.3.7 que $M^{1,\infty}(X) \subseteq D^\infty(X) = N^{1,\infty}(X)$. Es más, veremos a través de algunos ejemplos (III.3.8) que en general existen espacios métricos X para los cuales $M^{1,\infty}(X) \subsetneq D^\infty(X) = N^{1,\infty}(X)$.

Los principales resultados de los Capítulos II y III han sido recogidos en el trabajo [36].

La desigualdad de Poincaré (1) que detallamos a continuación es una definición ya bien establecida dentro del campo del análisis en espacios métricos de medida que fue introducida en [59, 58]. Sea $1 \leq p < \infty$. Diremos que (X, d, μ) admite una *desigualdad p-Poincaré* si existen constantes $C_p > 0$ y $\lambda \geq 1$ tales que para cada función medible Borel $f : X \to \mathbb{R} \cup \{\infty, -\infty\}$ y cada gradiente superior $g : X \to [0, \infty]$ de f, el par (f, g) satisface la desigualdad

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq C_p \, r \Big(\fint_{B(x,\lambda r)} g^p d\mu \Big)^{1/p} \tag{1}$$

para cada bola $B(x, r) \subset X$. Esta desigualdad de suele llamar "débil" porque se admite la posibilidad de que λ sea estrictamente mayor que 1.

En esta desigualdad, $B(x, r)$ denota la bola abierta de centro x y radio $r > 0$. Para un conjunto arbitrario $A \subset X$ con $0 < \mu(A) < \infty$ escribimos

$$f_A = \fint_A f = \frac{1}{\mu(A)} \int_A f \, d\mu.$$

Existe una amplia variedad de espacios métricos que admiten una desigualdad de p-Poincaré, que incluye ejemplos muy conocidos tales como $\mathbb{R}^n$, las variedades Riemannianas con curvatura de Ricci no negativa, los grupos de Carnot (en particular el grupo de Heisenberg), y también otros espacios métricos de medida no-Riemannianos con dimensión de Hausdorff fraccional, como puede verse por ejemplo en [80], [55], [70]; véanse también las referencias que aparecen en dichos trabajos. Un hecho sorprendente es que algunas de las consecuencias geométricas de esta condición son aparentemente independientes del parámetro p y la situación no está por el momento del todo clara. Este hecho se puede apreciar por ejemplo en [70] (condición de $\mathrm{Lip} - \mathrm{lip}$), [94] (casi-convexidad), o [24] (estructuras diferenciables medibles y persistencia de la desigualdad de Poincaré bajo límites Gromov-Hausdorff puntuales de espacios métricos).

Se sigue de la desigualdad de Hölder que si un espacio admite una desigualdad p-Poincaré entonces admite una desigualdad q-Poincaré para cada $q \geq p$. Recientemente, Keith y Zhong [73] han probado una propiedad de auto-mejora para las desigualdades de Poincaré, esto es, si X es un espacio métrico completo dotado de una medida doblante y que satisface una desigualdad p-Poincaré para algún $1 < p < \infty$, entonces existe $\varepsilon > 0$ tal que X admite una desigualdad q-Poincaré para todo $q > p - \varepsilon$. La más restrictiva de dichas desigualdades sería la desigualdade 1-Poincaré, y es bien conocido el hecho de que la desigualdad 1-Poincaré es equivalente a una propiedad isoperimétrica [87], [13].

Una pregunta que surge de manera natural es cuál sería la versión más débil de la desigualdad p-Poincaré que aún proporciona suficiente información geométrica sobre un espacio métrico. El Capítulo IV está dedicado a presentar nuestros resultados sobre el estudio de desigualdades p-Poincaré en el caso límite $p = \infty$ (véase Definición IV.1.1). Una de las implicaciones geométricas más útiles hasta el momento de la desigualdad p-Poincaré para p finito es el hecho de que los espacios métricos completos doblantes y que admiten una desigualdad p-Poincaré son casi-convexos (véase [94] o [54]). Si X admite una desigualdad ∞-Poincaré, la conclusión es la misma como muestra la Proposición IV.1.5. Sin embargo, como se muestra en el Corolario IV.2.16, la casi-convexidad no es una condición suficiente para que un espacio admita una desigualdad ∞-Poincaré y por tanto introduciremos la noción más fuerte de "thick" casi-convexidad (Definición IV.2.1). Un espacio métrico de medida se dice *"thick" casi-convexo* si todo par de conjuntos de medida positiva, separados una distancia positiva, se pueden conectar por una familia "gruesa" de curvas casi-convexas en el sentido de que el ∞-módulo de dicha familia de curvas es positivo. Este nuevo concepto geométrico nos permite probar en el Teorema IV.2.8 una caracterización geométrica en términos del ∞-módulo de curvas del espacio y también a una caracterización puramente analítica que pone en juego diferentes tipos de espacios de Lipschitz y espacios de Sobolev en el contexto de espacios métricos de medida. Para ser exactos, mostraremos que un espacio métrico doblante y completo admite una desigualdad ∞-Poincaré si y sólo si es "thick" casi-convexo, que es una condición puramente geométrica. Probaremos también que dicha condición es equivalente a la condición analítica de que $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ con seminormas de energía comparables, es decir, toda función de Lipschitz pertenece a una clase de equivalencia de $N^{1,\infty}(X)$, y toda función que pertenece a una clase de equivalencia de $N^{1,\infty}(X)$ se puede modificar en un conjunto de medida cero para convertirse en una función de Lipschitz y además los espacios tienen seminormas de energía comparables.

En este capítulo señalaremos también algunas de las diferencias entre las consecuencias de la desigualdad p-Poincaré y las de la desigualdad ∞-Poincaré. Estas diferencias aparecen por el hecho de que la norma L^∞ no es sensible a pequeñas perturbaciones locales.

Además, estudiaremos un concepto análogo a la "thick" casi-convexidad asociado con la desigualdad p-Poincaré para $p \geq 1$ finito, *p-thick casi-convexidad* (ver Definición IV.2.1) y probaremos en la Proposición IV.3.5 que los espacios que admiten una desigualdad p-Poincaré son p-thick casi-convexos. Daremos también un ejemplo (Ejemplo IV.3.6) que muestra como esta condición geométrica (p-thick casi-convexidad) no implica la validez de una desigualdad p-Poincaré. El espacio métrico de medida que construimos en este ejemplo es un espacio dotado de una medida doblante, que admite una desigualdad ∞-Poincaré, pero que no admite ninguna desigualdad p-Poincaré para p finito. Por tanto este ejemplo pone de manifiesto además que no se puede esperar una propiedad de auto-mejora de la desigualdad ∞-Poincaré del estilo de Keith y Zhong [73].

Acabaremos el Capítulo IV tratando el problema de la persistencia de la desigualdad ∞-Poincaré bajo la convergencia de Gromov-Hausdorff. La discusión del Capítulo 9 en [24] prueba que si $\{X_n, d_n, \mu_n\}_n$ es una sucesión de espacios métricos de medida con μ_n medidas doblantes y que admiten una desigualdad p-Poincaré, estando las constantes asociadas a la condición doblante y a la desigualdad de Poincaré uniformemente acotadas, y además, la sucesión de espacios métricos de medida converge en el sentido Gromov-Hausdorff a un espacio métrico de medida (X, d, μ), entonces el espacio límite es también doblante y admite una desigualdad p-Poincaré. En esta última parte del capítulo daremos un ejemplo (Ejemplo IV.3.9) que muestra que la persistencia de la desigualdad ∞-Poincaré falla bajo la convergencia de Gromov-Hausdorff.

En la discusión anterior se ha puesto de manifiesto que, para obtener un marco apropiado para el tipo de cálculo que estamos manejando, necesitamos que nuestro espacio no sólo tenga curvas rectificables, sino múltiples curvas distribuidas de manera uniforme a todas las escalas. Es bien conocido que algunos fractales clásicos, como la alfombra de Sierpiński (y el triángulo de Sierpiński) tienen curvas rectificables (de hecho curvas casi-convexas), pero no en suficiente cantidad para nuestros objetivos; esto es, en términos del módulo y de las desigualdades de Poincaré (véase el Ejemplo IV.2.16 y la discusión en [94, 2.3]).

En los trabajos [37] y [38] se han recogido los principales resultados del Capítulo IV.

En los últimos años la geometría fractal se ha desarrollado rápidamente en el seno de la teoría geométrica de la medida, el análisis armónico, los sistemas dinámicos y la teoría ergódica. Por ejemplo, se puede construir un operador análogo al Laplaciano en fractales para tratar problemas relacionados con el transporte continuo como la conducción del calor (véase [101] y las referencias que allí aparecen). En estas líneas, también el movimiento Browniano en la alfombra de Sierpiński ha atraído mucho interés en los últimos años [9].

Nosotros entenderemos por *alfombra* un espacio métrico homeomorfo a la alfombra de Sierpiński S_3 (véase definición en IV.2.15). Un problema fundamental en el estudio de las funciones casi-conformes y de las funciones bi-Lipschitz entre alfombras consiste en caracterizar las curvas rectificables contenidas en una alfombra.

Por ejemplo, dicha caracterización podría servir para dar una prueba directa de la siguiente propiedad de rigidez bi-Lipschitz de $S_{\mathbf{3}}$: toda aplicación bi-Lipschitz de $S_{\mathbf{3}}$ en sí misma es la restricción de una isometría del plano que conserva el cuadrado unidad. La rigidez bi-Lipschitz de $S_{\mathbf{3}}$ es un corolario de la rigidez casi-simétrica, que fue establecida por Bonk y Merenkov [20] usando técnicas conformes del módulo. Hasta donde sabemos, no existe ninguna prueba independiente de la rigidez bi-Lipschitz que no use técnicas conformes del módulo. Más resultados sobre la geometría conforme de las alfombras se pueden encontrar en [71], [19], [17], [86], [83]. Queremos resaltar además que la geometría conforme de la alfombra surge en conexión con la conjetura de Kapovich–Kleiner sobre uniformización casi-simétrica de las "group boundaries" de la alfombra de Sierpiński. Para más detalles, se puede consultar [18].

Toda curva rectificable contenida en la alfombra de Sierpiński es en particular una curva rectificable del plano y por tanto, por el teorema Rademacher, existe una recta tangente en casi todo punto de dicha curva. Por ello comenzamos de manera natural nuestro estudio considerando los segmentos contenidos en dichas alfombras. El objetivo del Capítulo V es dar una descripición completa del conjunto de las pendientes de los segmentos contenidos en una cierta clase de alfombras. De hecho, caracterizaremos el conjunto de pendientes de segmentos no-triviales contenidos en las alfombras de Sierpiński auto-semejantes (Teorema V.2.1 y Teorema V.2.10). Además, relacionaremos el conjunto de pendientes con los números de Farey (Proposición V.2.15) y con la dinámica de los llamados " billares tóricos punteados" (Observación V.2.11). Como consecuencia, deduciremos en la Proposición V.3.1 conclusiones sobre la colección de curvas diferenciables en todo punto contenidas en dichas alfombras. Nuestros resultados pueden ser considerados como un primer paso hacia la obtención de una descripción completa de las curvas rectificables contenidas en dichas alfombras.

Los resultados del Capítulo V han sido recogidos en [39].

Hemos incluido también un apéndice (Capítulo VI) dedicado a las propiedades de diferenciabilidad de las *funciones $\mathcal{H}$-Lipschitz* definidas sobre los *espacios abstractos de Wiener* y con valores en espacios métricos. Como ya hemos mencionado antes, el teorema clásico de Rademacher asegura que toda función Lipschitz f de $\mathbb{R}^n$ a $\mathbb{R}^k$ es Frechét diferenciable en casi todo punto con respecto a la medida de Lebesgue. Sin embargo, este resultado no tiene generalizaciones directas al caso infinito dimensional por dos razones principales. La primera reside en el hecho de que no existe un análogo infinito dimensional de la medida de Lebesgue. El segundo es el hecho de que existen funciones de Lipschitz definidas entre espacios de Hilbert, que no tienen ningún punto de diferenciabilidad Frechét. Por otra parte, si consideramos aplicaciones cuyo dominio de llegada es un espacio métrico, las propiedades de diferenciabilidad no se pueden interpretar en términos clásicos.

Comenzaremos la Sección VI.1 recordando qué se entiende por una *medida Gaussiana*. A continuación, daremos en la Sección VI.2 algunas definiciones básicas relacionadas con la estructura de espacio de Wiener. Después definiremos en la Sección VI.3 las *funciones $\mathcal{H}$-Lipschitzianas* y las compararemos con una cierta clase de funciones de Sobolev definidas a partir de una medida Gaussiana. Finalmente, en la Sección VI.4 recordaremos los conceptos de *diferenciabilidad métrica* y w^*-*diferenciabilidad* y terminaremos dando un teorema de Rademacher en este contexto.

Los resultados presentados en el Capítulo VI han sido recogidos en el trabajo [2].

Chapter I

Analysis on metric spaces: Introduction and Preliminaries

I.1 Introduction

Recent years have seen many advances in geometry and analysis, where first order differential calculus has been extended from the classical Euclidean setting to the setting of spaces with no a priori smooth structure; for a general introduction to the subject we mention here the survey works by Heinonen [55, 56], Ambrosio-Tilli [4], Hajłasz-Koskela [54] or Semmes [94]. The study of analysis on *metric measure spaces*, that is, metric spaces equipped with a measure, has in addition found many applications in several areas of analysis, such as nonlinear potential theory [10],[11],[97],[76], Carnot groups [23],[84],[7], the theory of quasiconformal and quasiregular mappings [59],[58],[57], non embedding results [88],[26], fractal analysis [94],[32],[101] or analysis on graphs [30]. See also [54] and [56] and the references therein.

In the late 1970s it had become clear that much of the basic analysis which involves functions only (and not their derivatives), can be done in (quasi-) metric spaces equipped with a *doubling* Borel *measure* (spaces of homogeneous type); see for example [28, 29]. A nontrivial Borel measure on a metric space is said to be *doubling* if the measure of a ball controls the measure of its double in a uniform manner. This condition imposed on the measure allows us to define the Hardy-Littlewood maximal operator or to talk about Lebesgue points and covering theorems of Vitali type.

However, the structure of a doubling metric measure space has turned out to be too weak to develop a first order differential calculus involving derivatives and extra hypothesis are needed. One can show for example that the lack of curves in many of the classical fractal sets is an obstacle for giving a reasonable notion of derivative. A useful feature of the Euclidean n-space, $n \geq 2$, is the fact that every pair of points x and y can be joined not only by the line segment $[x, y]$, but also by a large family of curves whose length is comparable to the distance between the points. Once one has found such a "thick" family of curves, the deduction of important Sobolev and Poincaré inequalities is an abstract procedure in which the Euclidean structure no longer plays a role. Up to now, the principal new requirement to the metric measure space for developing a first order calculus is the validity of a *p-Poincaré inequality*, which creates a link between the measure, the metric and the (length of the) gradient. Moreover, it provides a way to pass from the infinitesimal information which gives the gradient to larger scales. Metric spaces with doubling measure and p-Poincaré inequality (see definition I.1) admit first order differential calculus akin to that in Euclidean spaces. More precisely, from Cheeger's work [24] (see also Keith [70]), metric spaces endowed with a doubling measure and supporting a p-Poincaré inequality admit a *measurable differentiable structure* for which Lipschitz functions can be differentiated almost everywhere. The existence of such a structure is an example of a geometric restriction that spaces supporting a p-Poincaré inequality must satisfy. A crucial aspect of Cheeger's work is a careful analysis of the infinitesimal behavior of Lipschitz functions.

Given a metric space (X, d), a function $f : X \to \mathbb{R}$ is *C-Lipschitz* if there exists a constant $C > 0$ such that

$$|f(x) - f(y)| \leq Cd(x, y),$$

for each $x, y \in X$. It has been realized for a long time that for many aspects of Euclidean analysis the best framework is that of Lipschitz functions. The usefulness of the concept of Lipschitz function becomes more remarkable in settings where it is not meaningful to speak about "smooth functions". Lipschitz functions are the natural substitute for smooth functions to be considered in a metric space. Actually, in the Euclidean setting, Rademacher's theorem [90] states that Lipschitz continuous functions are differentiable almost everywhere with respect to the Lebesgue measure. As one can appreciate, the Lipschitz condition is a purely geometric condition that makes perfect sense in the metric setting and gives *global information* about the space.

On the other hand, the notion of derivative yields *infinitesimal information*: it measures the infinitesimal oscillations of a function at a given point. However, a metric space is not necessarily endowed with a natural linear or differentiable structure and one does not have a derivative, even in the weak sense of Sobolev spaces. Nevertheless, if f is a real-valued function on a metric space (X, d) and x is a point in X, one can use similar measurements of sizes of first-order oscillations of f at small scales around x, such as

$$D_r f(x) = \frac{1}{r} \sup \Big\{ |f(y) - f(x)| : y \in X, d(x, y) \leq r \Big\}.$$

Although this quantity does not contain as much information as standard derivatives on Euclidean spaces do (since we omit the signs), it makes sense in more general settings because we do not need any special behavior of the underlying space to define it. In fact, if we look at the superior limit of the above expression as r tends to 0 we almost recover in many cases, as in the Euclidean or Riemannian setting, the standard notion of modulus of the derivative. More precisely, given a continuous function $f : X \to \mathbb{R}$, the *pointwise Lipschitz constant* at a point $x \in X$ is defined as follows:

$$\operatorname{Lip} f(x) = \limsup_{r \to 0} D_r f(x) = \limsup_{\substack{y \to x \\ y \neq x}} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Recently, this functional has played an important role in several contexts. Heinonen and Koskela [59, 60] introduced the notion of "upper gradients" which serves the role of derivatives in a metric space X. A nonnegative Borel function g on X is said to be an *upper gradient* for an extended real-valued function f on X if

$$|f(\gamma(a)) - f(\gamma(b))| \leq \int_\gamma g,$$

for every rectifiable curve $\gamma : [a, b] \to X$. Observe that the pointwise Lipschitz function $\operatorname{Lip} f$ is an upper gradient for any Lipschitz function f. We also mention here the construction of differentiable structures in the setting of metric measure spaces [24],[70], and the Stepanov differentiability theorem [6].

This concept gives rise to the *pointwise Lipschitz function spaces*, which contain in some sense infinitesimal information about the functions. More precisely we will consider the following function spaces

- $\diamond$ $D(X) = \{f : X \longrightarrow \mathbb{R} : \|\operatorname{Lip} f\|_\infty < +\infty\}$
- $\diamond$ $D^\infty(X) = \{f : X \longrightarrow \mathbb{R} : \|\operatorname{Lip} f\|_\infty < +\infty \text{ and } \|f\|_\infty < \infty\}$.

The space $D(X)$ (respectively $D^\infty(X)$) clearly contains the space $\mathrm{LIP}(X)$ of Lipschitz functions (respectively the space of bounded Lipschitz functions $\mathrm{LIP}^\infty(X)$). One interesting problem is to know under which circumstances one can use information that is known infinitesimally, to yield information that holds globally throughout the space.

One of the aims of this thesis is to present our results in understanding the infinitesimal versus global behavior of Lipschitz functions in the metric setting. This will be done in Chapter II. The key assumption needed is that the space where the map is defined should be highly connected, meaning that there are many paths joining any part of the space. More precisely, in Corollary II.1.4 we give sufficient conditions on the metric space X to guarantee the equality between $D(X)$ and $\mathrm{LIP}(X)$. In particular, we obtain a positive result for the class of quasiconvex spaces. A space is *quasiconvex* if there exists a constant such that every pair of points can be connected with a curve whose length is at most the constant times the distance between the points. These spaces will be very useful when proving a partial converse of Corollary II.1.4. In addition, we present some examples for which $\mathrm{LIP}(X) \neq D(X)$ (see Examples II.1.5 and II.1.6).

At this point, it seems natural to approach the problem of determining which kind of spaces can be classified by their pointwise Lipschitz structure. Our strategy will be to follow the proof in [47] where the authors find a large class of metric spaces for which the algebra of bounded Lipschitz functions determines the Lipschitz structure for X. A crucial point in the proof is the use of the Banach space structure of $\mathrm{LIP}(X)$. Thus, we endow $D(X)$ with a norm which arises naturally from the definition of the operator Lip. This norm is not complete in the general case, as it can be seen in Example II.2.4. However, Theorem II.2.3 states that there is a wide class of spaces, the *locally radially quasiconvex metric spaces* (see Definition II.2.1), for which $D^\infty(X)$ admits the desired Banach space structure. Moreover, for such spaces, we obtain a kind of Banach-Stone theorem in this framework (see Theorem II.3.6).

We also deal with *pointwise isometries* between metric spaces, related to pointwise Lipschitz functions. In Lemma II.4.3 we prove that given (X, d_X) and (Y, d_Y) complete locally radially quasiconvex metric spaces there exists an isomorphism of vector lattices between $D^\infty(X)$ and $D^\infty(Y)$ which is an isometry for the respective norms if and only if X and Y are pointwise isometric. It is clear that if two metric spaces are locally isometric, then they are pointwise isometric. The converse is not true in general, as Example II.4.4 shows.

We further study in Corollaries II.2.10 and II.2.11 the problem of characterizing when the function space of (bounded) pointwise Lipschitz functions admits a *Banach linearization.* Recall that linearization is a useful tool for studying function spaces, since it enables the application of linear functional analysis to problems concerning nonlinear functions. We also show in Example II.2.12 that there exist metric spaces (X, d) for which $D^\infty(X)$ is a dual space but it does not admit a Banach linearization over X.

If we have a Borel measure on the metric space, we can deal with many more problems. In this line, there are for example generalizations of classical Sobolev spaces to the setting of arbitrary metric measure spaces. Hajłasz was the first who introduced Sobolev type spaces in this context [52]. He defined the spaces $M^{1,p}(X)$ for $1 \leq p \leq \infty$ in connection with maximal operators. It is well known that $M^{1,\infty}(X)$ is in fact the space of bounded Lipschitz functions on X. Shanmugalingam in [96] introduced, using the notion of upper gradient (and more generally weak upper gradient), the *Newtonian spaces* $N^{1,p}(X)$ for $1 \leq p < \infty$. For further information about different types of Sobolev spaces on metric measure spaces see [53]. In Chapter III of the thesis, we will generalize the definition of *Newtonian spaces* to the case $p = \infty$ and we will study the main properties of the space $N^{1,\infty}(X)$. To be able to introduce the space $N^{1,\infty}(X)$ we will need first the corresponding definitions and main properties of ∞-*modulus of a family of curves* (see Definition III.2.1) and ∞-*weak upper gradients* (see Definition III.2.7). We further define the ∞-*capacity* (see Definition III.2.13), a useful ingredient when proving that $N^{1,\infty}(X)$ is a Banach space. This will be done in Theorem III.2.17.

One of the main results of Chapter III is Theorem III.3.3. It enables us to construct, for the class of doubling metric spaces with a p-Poincaré inequality, quasiconvex curves which avoid zero measure sets. This is a technical result which allows us to compare the function spaces $D^\infty(X)$ and $\mathrm{LIP}^\infty(X)$ with the Sobolev space, $N^{1,\infty}(X)$. In particular, for metric spaces with a doubling measure and a p-Poincaré inequality we prove in Corollary III.3.5 the equality of all the mentioned spaces. Furthermore, if we just require a uniform local p-Poincaré inequality we obtain in Theorem III.3.7 that $M^{1,\infty}(X) \subseteq D^\infty(X) = N^{1,\infty}(X)$.

Moreover, we will see throughout some examples (III.3.8) that there exist metric spaces X for which $M^{1,\infty}(X) \subsetneq D^{\infty}(X) = N^{1,\infty}(X)$.

The main results in Chapters II and III have given rise to the publication [36].

The following Poincaré inequality is now standard in literature on analysis in metric measure spaces. An abstract Poincaré inequality in a metric measure space, as discussed here, was formulated in [59, 58]. Let $1 \leq p < \infty$. We say that (X, d, μ) supports a *weak p-Poincaré inequality* if there exist constants $C_p > 0$ and $\lambda \geq 1$ such that for every Borel measurable function $f : X \to \mathbb{R} \cup \{\infty, -\infty\}$ and every upper gradient $g : X \to [0, \infty]$ of f, the pair (f, g) satisfies the inequality

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq C_p \, r \Big(\fint_{B(x,\lambda r)} g^p d\mu \Big)^{1/p} \tag{I.1}$$

for each ball $B(x, r) \subset X$. The word *weak* refers to the possibility that λ may be strictly greater than 1.

Here $B(x, r)$ is an open ball with center at x and radius $r > 0$. For arbitrary $A \subset X$ with $0 < \mu(A) < \infty$ we write

$$f_A = \fint_A f = \frac{1}{\mu(A)} \int_A f \, d\mu.$$

There is a long list of metric spaces supporting a Poincaré inequality, including some standard examples such as $\mathbb{R}^n$, Riemannian manifolds with nonnegative Ricci curvature, Carnot groups (in particular the Heisenberg group), but also other nonRiemannian metric measure spaces of fractional Hausdorff dimension, see for example [80], [55], [70] and references therein. One surprising fact is that some geometric consequences of this condition seem to be independent of the parameter p and the picture is not yet clear. This fact can be appreciated for example in [70] ($\mathrm{Lip} - \mathrm{lip}$ condition), [94] (quasiconvexity), or [24] (measurable differentiable structure and persistence of Poincaré inequality under pointed measured Gromov-Hausdorff limits of metric spaces).

It follows from Hölder's inequality that if a space admits a p-Poincaré inequality, then it admits a q-Poincaré inequality for each $q \geq p$. Recently Keith and Zhong [73] proved a self-improving property for Poincaré inequalities, that is, if X is a complete metric space equipped with a doubling measure satisfying a p-Poincaré inequality for some $1 < p < \infty$, then there exists $\varepsilon > 0$ such that X supports a q-Poincaré inequality for all $q > p - \varepsilon$. The strongest of all these inequalities would be a 1-Poincaré inequality, and it is well known that the 1-Poincaré inequality is equivalent to the relative isoperimetric property [87], [13]. A natural question is what would be the weakest version of p-Poincaré inequality that would still give reasonable information on the geometry of the metric space.

Chapter IV is devoted to present our results in the study of p-Poincaré inequalities for the limit case $p = \infty$ (see Definition IV.1.1). One of the most useful geometric implications of the p-Poincaré inequality for finite p is the fact that if a complete doubling metric measure space supports a p-Poincaré inequality then the space is quasiconvex (see [94] or [54]). If X is only known to support an ∞-Poincaré inequality, the same conclusion holds as demonstrated by Proposition IV.1.5. However, as one can appreciate in Corollary IV.2.16, quasiconvexity is not a sufficient condition for a space to support an ∞-Poincaré inequality so we will introduce the stronger notion of thick quasiconvexity (Definition IV.2.1). A metric measure space is said to be *thick quasiconvex* if, loosely speaking, every pair of sets of positive measure, which are a positive distance apart, can be connected by a "thick" family of quasiconvex curves in the sense that the ∞-modulus of this family of curves is positive. This new geometric concept leads us in Theorem IV.2.8 to obtain a geometric characterization in terms of ∞-modulus of curves in the space and also a purely analytic condition which interplays different Lipschitz-type function spaces and Sobolev spaces in the setting of arbitrary metric measure spaces. More precisely, we show that a connected complete doubling metric measure space supports a weak ∞-Poincaré inequality if and only if it is thick quasiconvex, which is a purely geometric condition. We will also prove that this condition is equivalent to the purely analytic condition that $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ with comparable energy seminorms, that is, every Lipschitz function belongs to an equivalence class in $N^{1,\infty}(X)$, every function in any equivalence class in $N^{1,\infty}(X)$ can be modified on a set of measure zero to become a Lipschitz continuous function and the spaces have comparable energy seminorms.

We will also point out some of the differences between the consequences of p-Poincaré inequality and that of ∞-Poincaré inequality. These differences appear to be due to the fact that unlike the L^p-norm for finite p, the L^∞-norm is not sensitive to small local perturbations. We study a concept analogous to thick quasiconvexity associated with p-Poincaré inequality for finite $p \geq 1$, *p-thick quasiconvextiy* (see Definition IV.2.1) and prove in Proposition IV.3.5 that spaces supporting a p-Poincaré inequality are p-thick quasiconvex. We also give an example (Example IV.3.6) which illustrates that this analogous geometric property (p-thick quasiconvexity) does not imply the validity of a p-Poincaré inequality. The metric measure space given in this example is doubling and supports an ∞-Poincaré inequality, but supports no finite p-Poincaré inequality. So this example shows in addition that one cannot expect a self-improving property for ∞-Poincaré inequalities in the spirit of Keith and Zhong [73].

We will finish Chapter IV discussing the persistence of ∞-Poincaré inequalities under Gromov-Hausdorff convergence. The discussion in Chapter 9 of [24] demonstrates that if $\{X_n, d_n, \mu_n\}_n$ is a sequence of metric measure spaces with μ_n doubling measures supporting a p-Poincaré inequality, and in addition the constants associated with the doubling property and Poincaré inequality are uniformly bounded and, furthermore, this sequence of metric measure spaces converges in the measured Gromov-Hausdorff sense to a metric measure space (X, d, μ), then this limit space also is doubling and supports a p-Poincaré inequality. We will provide an example (Example IV.3.9) which demonstrates that this persistence of Poincaré inequality under measured Gromov-Hausdorff limits fails for ∞-Poincaré inequality.

The main results of Chapter IV have been collected in the works [37] and [38].

From the previous discussion it has become clear that, to obtain a setting where the type of calculus we are looking at is possible, we need a space which not only has rectifiable curves, but also plenty of them uniformly at all scales. It is known that some classical fractals, such as the Sierpiński carpets (and Sierpiński Gaskets) have rectifiable curves (they are indeed quasiconvex), but they are not enough for our purposes; that is, in terms of modulus and Poincaré inequalities (see Example IV.2.16 and discussion in [94, 2.3]).

On the other hand, in the last years fractal geometry has developed quickly on the foundation of geometric measure theory, harmonic analysis, dynamical systems and ergodic theory. For example, one can construct an analogous operator to the Laplacian on fractals in order to deal with continuous transport problems like heat conduction (see [101] and references therein). Brownian motion on the Sierpiński carpet has also attracted interest in recent years [9].

A *carpet* is a metric space which is homeomorphic to the Sierpiński carpet $S_{\mathbf{3}}$ (see definition in IV.2.15). A fundamental problem in the study of quasiconformal and bi-Lipschitz maps between carpets is to characterize the rectifiable curves contained in a given carpet.

For instance, such a characterization could perhaps be used to give a direct proof of the following bi-Lipschitz rigidity property of $S_{\mathbf{3}}$: every bi-Lipschitz map of $S_{\mathbf{3}}$ onto itself is the restriction of an isometry of the plane which preserves the unit square. The bi-Lipschitz rigidity of $S_{\mathbf{3}}$ is a corollary of the quasisymmetric rigidity, which has been established by Bonk and Merenkov [20] using conformal modulus techniques. As far as we are aware, there is no independent proof of bi-Lipschitz rigidity which does not use conformal methods. Further results on the conformal geometry of carpets can be found in [71], [19], [17], [86], [83]. We remark that the conformal geometry of carpets arises in connection with the Kapovich–Kleiner conjecture on quasisymmetric uniformization of Sierpiński carpet group boundaries. See [18] for additional details.

Every rectifiable curve contained in a Sierpiński carpet is, in particular, a rectifiable curve in the plane and hence admits a tangent line at almost every point by the theorem of Rademacher [90]. We thus naturally begin by considering the line segments contained in such carpets. The aim of Chapter V is to give a complete description of the slopes of nontrivial line segments contained in the members of a class of square Sierpiński carpets. We will characterize the slopes of nontrivial line segments contained in self-similar Sierpiński carpets (Theorem V.2.1 and Theorem V.2.10). In addition, the set of slopes is related to Farey sequences (Proposition V.2.15) and the dynamics of punctured square toral billiards (Remark V.2.11). As a consequence, we deduce in Proposition V.3.1 conclusions about the collection of everywhere differentiable curves contained in such carpets.

These results provide a first step towards a description of the rectifiable curves contained in such carpets.

The results in Chapter V have been collected in the article [39].

We have also included an Appendix (Chapter VI) devoted to the differentiability properties of *$\mathcal{H}$-Lipschitz maps* defined on *abstract Wiener spaces* and with values in metric spaces. As mentioned before, the classical Rademacher theorem states that any Lipschitzian mapping f from $\mathbb{R}^n$ to $\mathbb{R}^k$ is Frechét differentiable almost everywhere, with respect to the Lebesgue measure. However, this result has no direct extensions to the infinite dimensional case for two main reasons. The first one, is the lack of infinite dimensional analogues of Lebesgue measure. The second one, is the existence of Lipschitzian mappings between Hilbert spaces that have no point of Frechét differentiability. On the other hand, if we consider a map taking values in a metric space, the differential properties cannot be interpreted in classical terms.

We start by recalling in Section VI.1 the concept of *Gaussian measure*. After that, we will give in Section VI.2 some basic definitions related to the Wiener space structure. We then define in Section VI.3 *$\mathcal{H}$-Lipschitzian maps* and compare it with some Sobolev classes over Gaussian measures. Finally, in Section VI.4 we recall the definition of *metric differentiability* and *w^*-differentiability* and we finish giving a Rademacher theorem in this context.

The results presented in Chapter VI have given rise to the publication [2].

I.2 Notation and Preliminaries

Let (X, d) be a metric space. For $x \in X$ and $r > 0$ we let $B(x, r) := \{y \in X : d(x, y) < r\}$ be the open ball of radius r centered at x. Analogously, $\overline{B}(x, r) := \{y \in X : d(x, y) \leq r\}$ is the closed ball of radius r around x. For $\lambda > 0$ we write $\lambda B(x, r)$ to mean $B(x, \lambda r)$. We point out here that in the abstract metric setting, while $\overline{B}(x, r)$ contains the closure of $B(x, r)$, it might be larger.

If A is a nonempty subset of X, then $\operatorname{diam} A$ denotes the *diameter* of A, defined by

$$\operatorname{diam} A := \sup\{d(x, y) : x, y \in A\}.$$

By a *curve* γ we will mean a continuous mapping $\gamma : [a, b] \to X$. The image of a curve will be denoted by $|\gamma| = \gamma([a, b])$. Recall that the *length* of a continuous curve $\gamma : [a, b] \to X$ in a metric space (X, d) is defined as

$$\ell(\gamma) := \sup\Big\{\sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1}))\Big\},$$

where the supremum is taken over all finite partitions $a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a, b]$. We will say that a curve γ is *rectifiable* if $\ell(\gamma) < \infty$. The *length function* associated with a rectifiable curve $\gamma : [a, b] \to X$ is $s_\gamma : [a, b] \to [0, \ell(\gamma)]$, given by $s_\gamma(t) = \ell(\gamma_{|[a,t]})$. The integral of a Borel function g over a rectifiable path γ is usually defined via the arc-length parametrization γ_0 of γ in the following way:

$$\int_\gamma g ds := \int_0^{\ell(\gamma)} g \circ \gamma_0(t) dt.$$

Recall here that every rectifiable curve γ admits a parametrization by the arc-length γ_0. More precisely, $\gamma = \gamma_0 \circ s_\gamma$ with $\gamma_0 : [0, \ell(\gamma)] \to X$ and $\ell(\gamma_{0|_{[0,t]}}) = t$ for all $t \in [0, \ell(\gamma)]$. Hence from now on we only consider curves that are arc-length parametrized. For a nice discussion about general facts of curves in metric spaces one can follows [53, Section 3].

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is *C-Lipschitz* if there exists a constant $C > 0$ such that

$$d_Y(f(x), f(y)) \leq C d_X(x, y),$$

for each $x, y \in X$. From now on, $\mathrm{LIP}(\cdot)$ will denote the *Lipschitz constant*:

$$\mathrm{LIP}(f) := \sup_{\substack{x,y \in X \\ x \neq y}} \frac{d_Y(f(x), f(y))}{d_X(x, y)}.$$

We denote by $\mathrm{LIP}^\infty(X,Y)$ the space of bounded Lipschitz functions from X to Y. If $Y=\mathbb{R}$ we simply denote it by $\mathrm{LIP}^\infty(X)$. The natural norm on $\mathrm{LIP}^\infty(X)$ is given by

$$\|f\|_{\mathrm{LIP}^\infty(X)} := \|f\|_\infty + \mathrm{LIP}(f).$$

Along this work, $\|\cdot\|_\infty$ will denote the supremum norm whereas $\|\cdot\|_{L^\infty}$ will denote the essential supremum norm, provided we have a measure on X.

A function is said to be *bi-Lipschitz* if it is Lipschitz and admits a Lipschitz inverse. An *isometry* is a 1-bi-Lipschitz map.

A metric space (X,d) is said to be a *length space* if for each pair of points $x,y\in X$ the distance $d(x,y)$ coincides with the infimum of all lengths of curves in X connecting x with y. Another interesting class of metric spaces, which contains length spaces, are the so called *quasiconvex* spaces. Recall that a metric space (X,d) is *quasiconvex* if there exists a constant $C\geq 1$ such that for each pair of points $x,y\in X$, there exists a curve γ connecting x and y with $\ell(\gamma)\leq Cd(x,y)$. As one can expect, a metric space is *quasiconvex* if, and only if, it is bi-Lipschitz homeomorphic to some length space.

A metric space is called *doubling* if there is a constant C so that every ball of radius r can be covered by at most C balls of radius $r/2$.

We can endow our metric spaces with a measure μ in which case, (X,d,μ) will denote a *metric measure space*, that is, a metric space equipped with a metric d and a *Borel regular* measure μ defined on the Borel sets $\mathcal{B}(X)$, that is, μ is an outer measure on (X,d) such that all Borel sets are μ-measurable and for each set $A\subset X$ there exists a Borel set B such that $A\subset B$ and $\mu(A)=\mu(B)$.

A measure μ is *doubling* if there is a constant $C_\mu\geq 1$ such that for all $x\in X$ and $r>0$,

$$0<\mu(B(x,2r))\leq C_\mu\,\mu(B(x,r))<\infty. \tag{I.2}$$

We shall denote by C_μ the least constant that satisfies condition (I.2), i.e., we define

$$C_\mu := \sup_B \frac{\mu(2B)}{\mu(B)}.$$

It is well-known that a complete metric space X admits a doubling measure if and only if X is doubling, see [82].

Examples I.2.1.

- If we take $X=\mathbb{R}^n$, $d(x,y)=|x-y|$ the Euclidean metric and $\mu=\mathscr{L}^n$ the Lebesgue measure, then $(\mathbb{R}^n,|\cdot|,\mathscr{L}^n)$ is a doubling metric measure space with $C_\mu=2^n$.
- Let (M,g) be a complete Riemannian manifold of dimension n and μ the canonical measure associated to the metric tensor g. Then, if the Ricci curvature is nonnegative it follows, from [25, Proposition 4] that μ is doubling with $C_\mu=2^n$.

An iteration of inequality (I.2) shows (see for example [4, Theorem 5.2.2]) that the doubling condition is equivalent to the existence of constants C and s_1 depending only on C_μ such that, whenever B is a ball in X, $x \in B$ and $r > 0$ with $B(x,r) \subset B$,

$$\frac{\mu(B(x,r))}{\mu(B)} \geq \frac{1}{C}\left(\frac{r}{\operatorname{rad}(B)}\right)^{s_1}, \tag{I.3}$$

where $\operatorname{rad}(B)$ denotes the radius of B. In this case μ is said to be s_1-*homogeneous*. It means that there exists a lower bound for the density of the space X. If, in addition, X is connected and has at least two points, then the doubling property also implies the existence of a constant $s_2 > 0$ such that for all balls $B \subset X$ and $B(x,r) \subset B$,

$$\frac{\mu(B(x,r))}{\mu(B)} \leq \frac{1}{C}\left(\frac{r}{\operatorname{rad}(B)}\right)^{s_2}. \tag{I.4}$$

Because of the above inequality, letting $r \to 0$ we see that for all $x \in X$ we have $\mu(\{x\}) = 0$, that is, μ has no atoms.

In a complete metric space X, the existence of a doubling measure which is not trivial and is finite on balls implies that X is separable and *proper*. The latter means that closed bounded subsets of X are compact. In particular, X is locally compact. Then, the notion of doubling metric spaces is intrinsically finite-dimensional; this implies that for example it is not possible to endow a priori infinite dimensional Banach spaces with doubling measures.

Some of the classical theorems in analysis in the Euclidean setting can be extended to doubling metric measure spaces. The Lebesgue Differentiation Theorem is such an example: if f is a locally integrable function on a doubling metric space X, then

$$f(x) = \lim_{r\to 0} \fint_{B(x,r)} f\,d\mu,$$

for μ-a.e. point in X. In other words, almost every point in X is a *Lebesgue point* for f; see for example [55, Theorem 1.8].

Here for arbitrary $A \subset X$ with $0 < \mu(A) < \infty$ we write

$$f_A = \fint_A f = \frac{1}{\mu(A)} \int_A f\,d\mu.$$

In particular, if A is a Borel set, the function $f = \chi_A$ is a locally integrable function on X and hence, μ-almost every point x in A is a *density point*, that is,

$$\lim_{r\to 0} \frac{\mu(B(x,r)\cap A)}{\mu(B(x,r))} = 1.$$

It is also possible to define the *maximal operator* and obtain the same continuity properties from $L^p(X,\mu)$ to $L^p(X,\mu)$ as in the Euclidean case (see [55, Theorem 2.2]). More precisely, there exists a constant C depending only on the doubling constant C_μ of X and on p such that for each $f \in L^1(X,\mu)$ and for all $t > 0$

$$\mu(\{M(f) > t\}) \leq \frac{C}{t}\int_X |f| d\mu, \tag{I.5}$$

and for each $f \in L^p(X,\mu)$

$$\int_X |M(f)|^p d\mu \leq C \int_X |f|^p d\mu.$$

Recall that

$$M(f)(x) = \sup_{r>0} \fint_{B(x,r)} |f| d\mu.$$

We also recall here the definition of *Hausdorff measure*. Let A be any subset of X, and $\delta > 0$ a real number. Define

$$H^s_\delta(A) = \inf\Big\{ \sum_{i=1}^{\infty} (\operatorname{diam} A_i)^s : \bigcup_{i=1}^{\infty} A_i \supset A, \operatorname{diam} A_i < \delta \Big\},$$

where the infimum is taken over all countable covers of A by sets $A_i \subset X$ satisfying $\operatorname{diam} A_i < \delta$. Observe that $H^s_\delta(A)$ is monotone decreasing in δ, so the following limit exists:

$$H^s(A) := \sup_{\delta>0} H^s_\delta(A) = \lim_{\delta>0} H^s_\delta(A).$$

It can be seen that $H^s(A)$ is an outer measure, and its resctriction to the σ-field of Caratheodory-measurable sets is a measure. It is called the *s-dimensional Hausdorff measure* of A. For a set $A \subset X$, the *Hausdorff dimension* of A is defined by

$$\dim_H(A) = \inf\{s \geq 0 : H^s(A) = 0\} = \sup\{s \geq 0 : H^s(A) = \infty\}.$$

Next we discuss the notion of *measured Gromov-Hausdorff convergence* of a sequence of metric measure spaces, $\{(X_n, d_n, \mu_n)\}_n$, to a metric measure space (X, d, μ). To that aim, we first recall the notion of *Gromov-Hausdorff convergence* between compact metric spaces. See [50] or [22, Ch. 7.4.] for further details.

Given a proper metric space (Z, d_Z), and two compact sets K_1, K_2 of Z, the *Hausdorff distance* $d_H(K_1, K_2)$ is the number

$$d_H(K_1,K_2) := \inf\left\{\varepsilon > 0 \,:\, K_2 \subset \bigcup_{z\in K_1} B(z,\varepsilon) \text{ and } K_1 \subset \bigcup_{z\in K_2} B(z,\varepsilon)\right\}.$$

A sequence of compact sets $\{K_n\}_n$ in Z is said to *converge in the Hausdorff topology* to a compact set $K \subset Z$ if $d_H(K_n, K) \to 0$ as $n \to \infty$. Given a sequence of proper metric subspaces $\{A_n\}_n$ of Z, $p \in A_n$ for all n, a proper metric subspace $A \subset Z$ and a point $p \in A$, we say that the pointed sequence $\{(A_n, p)\}_n$ converges in the Hausdorff topology to (A, p) if for all $r > 0$ the sequence of compact sets $\{\overline{B}(p, r) \cap A_n\}_n$ converges in the Hausdorff topology to $\overline{B}(p, r) \cap A$.

We next recall the *pointed measured Gromov-Hausdorff convergence.* The notion of *pointed measured Gromov-Hausdorff convergence* was introduced by Fukaya in [44]. See also [69], [24] and references therein.

Definition I.2.2. A sequence of proper pointed metric measure spaces $\{(X_n, d_n, \mu_n, p_n)\}$ is said to *converge* to another complete pointed metric measure space (X, d, μ, p) if there exists a proper pointed metric metric space (Z, ρ, q) and isometric embeddings $i : X \to Z$ and $i_n : X_n \to Z$ for each $n \in \mathbb{N}$ such that $i_n(p_n) = q = i(p)$, $(i_n(X_n), q)$ converges to $(i(X), q)$ in the above-mentioned sense of Hausdorff topology on Z, and such that $(i_n)_{\#}\mu_n$ converges to $i_{\#}\mu$ in the weak* sense.

In the above definition, $i_{n\#}\mu_n$ is the push-forward of the measure μ_n under the isometry i_n; for sets $A \subset Z$, we have $i_{n\#}\mu_n(A) = \mu_n(i_n^{-1}(A))$. We say that a sequence of Borel measures μ_n on Z *converges in the weak* sense* to a Borel measure μ if for all compactly supported continuous functions φ on Z,

$$\lim_{n\to\infty} \int_Z \varphi \, d\mu_n = \int_Z \varphi \, d\mu.$$

If for all Borel sets $A \subset Z$ we have $\mu_n(A) \to \mu(A)$ as $n \to \infty$, then μ_n converges in the weak* sense to μ, but the converse is not always true, as shown by the measures μ_n given by $d\mu_n = [1 - (1 - n^{-1})^2]^{-1} \chi_{B((0,0),1)\setminus B((0,0),1-n^{-1})} \, d\mathscr{L}^2$ and $\mu = (2\pi)^{-1} \mathcal{H}^1 | S^1((0, 0), 1)$ on $\mathbb{R}^2$.

If a sequence of compact sets $\{K_n\}_n$ of Z converges in the Hausdorff topology to a compact set $K \subset Z$, then this sequence converges in the Gromov-Hausdorff sense to K, but again the converse need not hold, as demonstrated by the example $Z = \mathbb{R}^2$, $K_n = \overline{B}((n, 0), 1)$, and $K = \overline{B}((0, 0), 1)$. The notion of Gromov-Hausdorff convergence is therefore more flexible and depends more on the shapes of the sequence of metric spaces approximating the shape of the limit space.

Unless otherwise stated, the letter C denotes various positive constants whose exact values are not important, and the value might change even from line to line.

Chapter II

Pointwise Lipschitz functions on metric spaces

As mentioned in the introduction, the study of analysis on metric measure spaces has progressed in recent years to include concepts from first order differential calculus [4],[55],[56],[94]. The notion of derivative measures the infinitesimal oscillations of a function at a given point, and gives information concerning for instance monotonocity. In general metric spaces we do not have a derivative, even in the weak sense of Sobolev spaces. Nevertheless, if f is a real-valued function on a metric space (X, d) and x is a point in X, one can use similar measurements of sizes of first-order oscillations of f at small scales around x, such as

$$D_r f(x) = \frac{1}{r} \sup \Big\{ |f(y) - f(x)| : y \in X, d(x,y) \leq r \Big\}.$$

On one hand, this quantity does not contain as much information as standard derivatives on Euclidean spaces do (since we omit the signs) but, on the other hand, it makes sense in more general settings because we do not need any special behavior of the underlying space to define it. In fact, if we look at the superior limit of the above expression as r tends to 0 we almost recover in many cases, as in the Euclidean or Riemannian setting, the standard notion of derivative. More precisely, given a continuous function $f : X \to \mathbb{R}$, the *pointwise Lipschitz constant* at a point $x \in X$ is defined as follows:

$$\operatorname{Lip} f(x) = \limsup_{r \to 0} D_r f(x) = \limsup_{\substack{y \to x \\ y \neq x}} \frac{|f(x) - f(y)|}{d(x,y)}.$$

Recently, this functional has played an important role in several contexts. We just mention here the construction of differentiable structures in the setting of metric measure spaces [24],[70], the theory of upper gradients [59],[95], or the Stepanov differentiability theorem [6].

This concept gives rise to a class of function spaces, *pointwise Lipschitz function spaces*, which contains in some sense infinitesimal information about the functions:

$$D(X) = \{ f : X \longrightarrow \mathbb{R} : \| \operatorname{Lip} f \|_\infty < +\infty \}.$$

This space $D(X)$ clearly contains the space $\mathrm{LIP}(X)$ of Lipschitz functions and a first approach will be comparing such spaces. In Section II.1 we will introduce *pointwise Lipschitz function spaces* $D(X)$ and we look for conditions regarding the geometry of the metric spaces we are working with in order to understand in which cases the pointwise

Lipschitz information yields the global Lipschitz behavior of a function. In addition, we present some examples for which $\mathrm{LIP}(X) \neq D(X)$ (see Examples II.1.5 and II.1.6).

At this point, it seems natural to approach the problem of determining which kind of spaces can be classified by their pointwise Lipschitz structure. Our strategy will be to follow the proof in [47] where the authors find a large class of metric spaces for which the algebra of bounded Lipschitz functions determines the Lipschitz structure for X. A crucial point in the proof is the use of the Banach space structure of $\mathrm{LIP}(X)$. Thus, in Section II.2, we endow $D(X)$ with a norm which arises naturally from the definition of the operator Lip. This norm is not complete in the general case, as it can be seen in Example II.2.4. However, Theorem II.2.3 states that there is a wide class of spaces, the *locally radially quasiconvex metric spaces* (see Definition II.2.1), for which $D^\infty(X)$ admits the desired Banach space structure. Moreover, for such spaces, we obtain in Section II.3 a kind of Banach-Stone theorem in this framework (see Theorem II.3.6). This structure will be very useful when proving a partial converse of Corollary II.1.4 (see Corollary II.2.8) and also for studying the problem of Banach linearization (II.2.10 and II.2.11). We also show in Example II.2.12 that there exist metric spaces (X, d) for which $D^\infty(X)$ is a dual space but it does not admit a Banach linearization over X. We finish this chapter dealing with *pointwise isometries*, a special kind of isometries related to pointwise Lipschitz functions (Section II.4).

II.1 Pointwise Lipschitz functions

Let (X, d) be a metric space. Given a function $f : X \to \mathbb{R}$, the *pointwise Lipschitz constant* of f at a non isolated point $x \in X$ is defined as follows:

$$\operatorname{Lip} f(x) = \limsup_{\substack{y \to x \\ y \neq x}} \frac{|f(x) - f(y)|}{d(x, y)}.$$

If x is an isolated point we define $\operatorname{Lip} f(x) = 0$. This value is also known as *upper scaled oscillation* (see [6],[5]) or as *pointwise infinitesimal Lipschitz number* (see [56]).

Examples II.1.1.

- If $f \in C^1(\Omega)$ where Ω is an open subset of Euclidean space, or of a Riemannian manifold, then $\operatorname{Lip} f = |\nabla f|$.
- Let $\mathbb{H}$ be the first Heisenberg group, and consider an open subset $\Omega \subset \mathbb{H}$. If $f \in C^1_H(\Omega)$, that is, f is H-continuously differentiable in Ω, then $\operatorname{Lip} f = |\nabla_H f|$ where $\nabla_H f$ denotes the horizontal gradient of f. For further details see [84].
- If (X, d, μ) is a metric measure space which admits a measurable differentiable structure $\{(X_\alpha, \mathbf{x}_\alpha)\}_\alpha$ and $f \in \mathrm{LIP}(X)$, then $\operatorname{Lip} f(x) = |d^\alpha f(x)|$ μ-a.e., where $d^\alpha f$ de-

notes the Cheeger's differential. For further information about measurable differentiable structures see [24],[70].

Loosely speaking, the operator $\operatorname{Lip} f$ estimates some kind of infinitesimal lipschitzian property around each point. Our first aim is to see under which conditions a function $f : X \to \mathbb{R}$ is Lipschitz if and only if $\operatorname{Lip} f$ is a bounded functional. It is clear that if f is a Lipschitz function, then $\operatorname{Lip} f(x) \leq \operatorname{LIP}(f)$ for every $x \in X$. More precisely, we consider the following spaces of functions:

- ⋄ $\operatorname{LIP}(X) = \{f : X \longrightarrow \mathbb{R} : f \text{ is Lipschitz}\}$
- ⋄ $D(X) = \{f : X \longrightarrow \mathbb{R} : \sup_{x\in X} \operatorname{Lip} f(x) = \|\operatorname{Lip} f\|_\infty < +\infty\}$.

We denote by $\operatorname{LIP}^\infty(X)$ (respectively $D^\infty(X)$) the space of bounded Lipschitz functions (respectively, bounded functions which are in $D(X)$) and $\mathcal{C}(X)$ will denote the space of continuous functions on X. It is not difficult to see that for $f \in D(X)$, $\operatorname{Lip} f$ is a Borel function on X and that $\|\operatorname{Lip}(\cdot)\|_\infty$ yields a seminorm in $D(X)$.

Since functions with uniformly bounded pointwise Lipschitz constant have a flavour of differentiability it seems reasonable to determine if the pointwise Lipschitz functions are in fact continuous. Namely,

Lemma II.1.2. *Let (X, d) be a metric space. Then $D(X) \subset \mathcal{C}(X)$.*

Proof. Let $x_0 \in X$ be a non isolated point and $f \in D(X)$. We are going to see that f is continuous at x_0. Since $f \in D(X)$ we have that $\|\operatorname{Lip} f\|_\infty = M < \infty$, in particular, $\operatorname{Lip} f(x_0) \leq M$. By definition we have that

$$\operatorname{Lip} f(x_0) = \inf_{r>0} \sup_{\substack{d(x_0,y)\leq r\\ y\neq x_0}} \frac{|f(x_0) - f(y)|}{d(x_0, y)}.$$

Fix $\varepsilon > 0$. Then, there exists $r > 0$ such that

$$\frac{|f(x_0) - f(z)|}{d(x_0, z)} \leq \sup_{\substack{d(x_0,y)\leq r\\ y\neq x_0}} \frac{|f(x_0) - f(y)|}{d(x_0, y)} \leq M + \varepsilon \qquad \forall z \in B(x_0, r),$$

and so

$$|f(x_0) - f(z)| \leq (M + \varepsilon)d(x_0, z) \qquad \forall z \in B(x_0, r).$$

Thus, if $d(x_0, z) \to 0$ then $|f(x_0) - f(z)| \to 0$, and so f is continuous at x_0. □

Now we look for conditions regarding the geometry of the metric space X under which $\operatorname{LIP}(X) = D(X)$ (respectively $\operatorname{LIP}^\infty(X) = D^\infty(X)$). As it can be expected, we need some kind of *connectedness*. In fact, we are going to obtain a positive answer in the class of *length spaces* or, more generally, of *quasiconvex spaces*.

We begin our analysis with a technical result.

Lemma II.1.3. *Let (X,d) be a metric space and let $f \in D(X)$ (or $f \in D^\infty(X)$). Let $x, y \in X$ and suppose that there exists a rectifiable curve $\gamma : [a,b] \to X$ connecting x and y, that is, $\gamma(a) = x$ and $\gamma(b) = y$. Then, $|f(x) - f(y)| \leq \|\operatorname{Lip} f\|_\infty \, \ell(\gamma)$.*

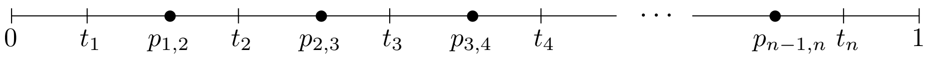
Proof. Since $f \in D(X)$, we have that $M = \|\operatorname{Lip} f\|_\infty < +\infty$. Fix $\varepsilon > 0$. For each $t \in [a,b]$ there exists $\rho_t > 0$ such that if $z \in B(\gamma(t), \rho_t) \setminus \{\gamma(t)\}$ then

$$|f(\gamma(t)) - f(z)| \leq (M+\varepsilon) d(\gamma(t), z).$$

Since γ is continuous, there exists $\delta_t > 0$ such that

$$I_t = (t - \delta_t, t + \delta_t) \subset \gamma^{-1}(B(\gamma(t), \rho_t)).$$

The family of intervals $\{I_t\}_{t \in [a,b]}$ is an open covering of $[a,b]$ and by compactness it admits a finite subcovering which will be denote by $\{I_{t_i}\}_{i=0}^{n+1}$. We may assume, refining the subcovering if necessary, that an interval I_{t_i} is not contained in I_{t_j} for $i \neq j$. If we relabel the indices of the points t_i in nondecreasing order, we can now choose a point $p_{i,i+1} \in I_{t_i} \cap I_{t_{i+1}} \cap (t_i, t_{i+1})$ for each $1 \leq i \leq n-1$. We might observe that such intersection is not empty because the family of intervals $\{I_{t_i}\}_{i=0}^{n+1}$ forms a chain. Thus, we obtain a sequence of points of the form:



Using the auxiliary points that we have just chosen, we deduce that:

$$d(x, \gamma(t_1)) + \sum_{i=1}^{n-1} \Big[d(\gamma(t_i), \gamma(p_{i,i+1})) + d(\gamma(p_{i,i+1}), \gamma(t_{i+1})) \Big] + d(\gamma(t_n), y) \leq \ell(\gamma),$$

and so $|f(x) - f(y)| \leq (M+\varepsilon)\ell(\gamma)$. Finally, since this is true for each $\varepsilon > 0$, we conclude that $|f(x) - f(y)| \leq \|\operatorname{Lip} f\|_\infty \, \ell(\gamma)$, as wanted. ☐

We say that $\mathrm{LIP}^\infty(X) = D^\infty(X)$ *with comparable energy seminorms* if the two sets are the same and there is a constant $C > 0$ such that for all $f \in \mathrm{LIP}^\infty(X)$,

$$\mathrm{LIP}(f) \leq C \|\operatorname{Lip} f\|_\infty.$$

As a straightforward consequence of the previous result, we deduce the following corollary.

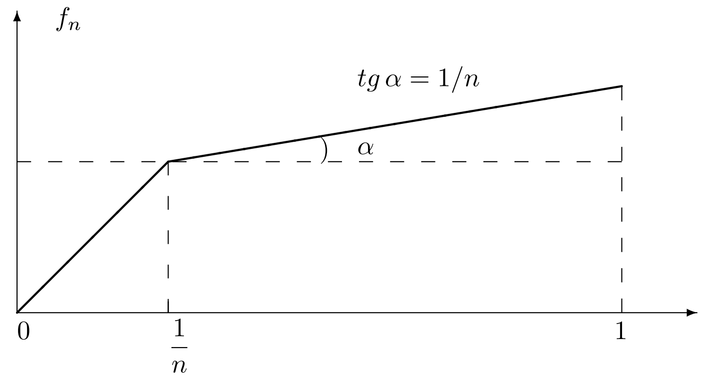
Corollary II.1.4. *If (X,d) is a quasiconvex space then* $\mathrm{LIP}^\infty(X) = D^\infty(X)$ *with comparable energy seminorms.*

The proof of the previous result is based on the existence of curves connecting each pair of points in X and whose length can be estimated in terms of the distance between the points. A reasonable kind of spaces in which we can approach the problem of determining if $\mathrm{LIP}(X)$ and $D(X)$ coincide, are the so called *chainable spaces*. It is an interesting class of metric spaces containing length spaces and quasiconvex spaces. Recall that a metric space (X,d) is said to be *well-chained* or *chainable* if for every pair of points $x, y \in X$ and for every $\varepsilon > 0$ there exists an ε-chain joining x and y, that is, a finite sequence of points $z_1 = x, z_2, \ldots, z_\ell = y$ such that $d(z_i, z_{i+1}) < \varepsilon$, for $i = 1, 2, \ldots, \ell - 1$. In such spaces there exist "chains" of points which connect two given points, and for which the distance between the *nodes*, which are the points $z_1, z_2, \ldots, z_\ell$, is arbitrarily small. However, there exist chainable spaces for which the spaces of functions $\mathrm{LIP}(X)$ and $D(X)$ do not coincide (see Example II.1.5).

Next, let us see throughout some examples that there exist complete metric spaces for which $\mathrm{LIP}(X) \subsetneq D(X)$. We will approach this by constructing two metric spaces for which $\mathrm{LIP}^\infty(X) \neq D^\infty(X)$. In the first example (Example II.1.5) we see that the equality fails "for large distances" while in the second one (Example II.1.6) it fails "for infinitesimal distances".

Example II.1.5. Define $X = [0,\infty) = \bigcup_{n\geq 1}[n-1, n]$, and write $I_n = [n-1, n]$ for each $n \geq 1$. Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ given by

$$f_n(x) = \begin{cases} x & \text{if } x \in \left[0, \frac{1}{n}\right] \\ \frac{nx+n-1}{n^2} & \text{if } x \in \left[\frac{1}{n}, 1\right]. \end{cases}$$



For each pair of points $x, y \in I_n$, we write $d_n(x,y) = f_n(|x-y|)$, and we define a metric on X as follows. Given a pair of points $x, y \in X$ with $x < y$, $x \in I_n$, $y \in I_m$ we define

$$d(x,y) = \begin{cases} d_n(x,y) & \text{if } n = m \\ d_n(x,n) + \sum_{i=n+1}^{m-1} d_i(i-1,i) + d_m(m-1,y) & \text{if } n < m. \end{cases}$$

A straightforward computation shows that d is in fact a metric and it coincides locally with the Euclidean metric d_e. More precisely,

$$\text{if } x \in I_n, \text{ on } J^x = \big(x - \frac{1}{n+1}, x + \frac{1}{n+1}\big) \text{ we have that } d|_{J^x} = d_e|_{J^x}.$$

Next, consider the bounded function $g : X \to \mathbb{R}$ given by

$$g(x) = \begin{cases} 2k - x & \text{if } x \in I_{2k}, \\ x - 2k & \text{if } x \in I_{2k+1}. \end{cases}$$

Let us check that $g \in D^\infty(X) \setminus \mathrm{LIP}^\infty(X)$. Indeed, let $x \in X$ and assume that there exists $n \geq 1$ such that $x \in I_n$. Then, we have that if $y \in J^x$,

$$\mathrm{Lip}\, f(x) = \limsup_{\substack{y \to x \\ y \neq x}} \frac{|g(x) - g(y)|}{d(x,y)} = \limsup_{\substack{y \to x \\ y \neq x}} \frac{|x-y|}{|x-y|} = 1.$$

Therefore, $g \in D^\infty(X)$.

On the other hand, for each positive integer n we have $|g(n-1) - g(n)| = 1$ and $d(n-1, n) = f_n(1) = \frac{2n-1}{n^2}$. Thus, we obtain that

$$\lim_{n\to\infty} \frac{|g(n-1) - g(n)|}{d(n-1,n)} = \lim_{n\to\infty} \frac{1}{\frac{2n-1}{n^2}} = \infty,$$

and so g is not a Lipschitz function.

In particular, since $\mathrm{LIP}(X) \neq D(X)$, we deduce by Corollary II.1.4 that X is not a quasiconvex space. Let us check straightforwardly that X is not quasiconvex.

Indeed, let $a, b \in \mathbb{N} \subset X$ such that $a \neq b$. We are going to compute the length of the curve

$$\gamma : [a,b] \to X, \ t \mapsto t$$

connecting these two points. Observe that taking a partition of the interval $[a,b]$, $a = t_0 < t_1 < \cdots < t_n = b$ in which the distance between two consecutive nodes is small enough, that is, such that for each i there exists $j \in \mathbb{N}$ such that $t_i, t_{i+1} \in [j, j+1]$ and $|t_i - t_{i+1}| < 1/j$, we obtain that

$$\ell(\gamma) = \sum_{i=0}^{n-1} d(t_i, t_{i+1}) = \sum_{i=0}^{n-1} |t_{i+1} - t_i| = |b - a|.$$

Observe that γ is clearly the shortest curve between all the ones connecting a to b. This is because the metric we have constructed coincides "infinitesimally" with the usual metric on $[0, +\infty)$.

Now, we are going to estimate the distance between a and b:

$$d(a,b) = \sum_{j=a+1}^{b} f_j(1) = \sum_{j=a+1}^{b} \frac{2}{j} - \frac{1}{j^2}.$$

Since $h : \mathbb{Z}^+ \to \mathbb{Q},\ n \mapsto h(n) = f_n(1)$ is a decreasing function we obtain that

$$\frac{2}{j} - \frac{1}{j^2} \leq \frac{2}{a} - \frac{1}{a^2} < \frac{2}{a},$$

and so

$$d(a,b) = \sum_{j=a+1}^{b} \frac{2}{j} - \frac{1}{j^2} \leq \sum_{j=a+1}^{b} \frac{2}{a} = \frac{2}{a}(b-a) = \frac{2}{a}\,\ell(\gamma). \tag{II.1}$$

Next, suppose by way of contradiction that there exists a constant $C > 0$ such that for each $x, y \in X$ there exists a curve γ in X joining x and y and so that $\ell(\gamma) \leq Cd(x,y)$.

Since γ is the shortest curve connecting a and b, we have

$$\frac{a}{2}\,d(a,b) \overset{(\text{II.1})}{\leq} \ell(\gamma) \leq Cd(a,b),$$

for each integer $a \geq 1$. Since $a \neq b$ we have $\frac{a}{2} \leq C$ for each integer $a \geq 1$, a contradiction. Thus, X is not quasiconvex.

However, X is a chainable space. Indeed, consider two points $x, y \in X$ and let $\varepsilon > 0$. Suppose that $x \in I_n$ e $y \in I_m$ and $n \leq m$. Take $\varepsilon' = \min\{\varepsilon, 1/m\}$ and consider the points $z_{s+1} = y$ and $z_k = x + k\varepsilon'$ with $0 \leq k \leq s$. We denote by s the integral part of the quotient $(y-x)/\varepsilon'$. We obtain that

$$d(z_k, z_{k+1}) \begin{cases} = \varepsilon' \leq \varepsilon & \text{if } 0 \leq k \leq s-1 \\ \leq \varepsilon' \leq \varepsilon & \text{if } k = s+1, \end{cases}$$

and so X is chainable. Hence X is a chainable space, for which $\mathrm{LIP}(X) \subsetneq D(X)$. $\square$

Example II.1.6. Consider the set

$$X = \{(x,y) \in \mathbb{R}^2 : y^3 = x^2, -1 \leq x \leq 1\} = \{(t^3, t^2), -1 \leq t \leq 1\},$$

and let d be the restriction to X of the Euclidean metric of $\mathbb{R}^2$. We define the bounded function

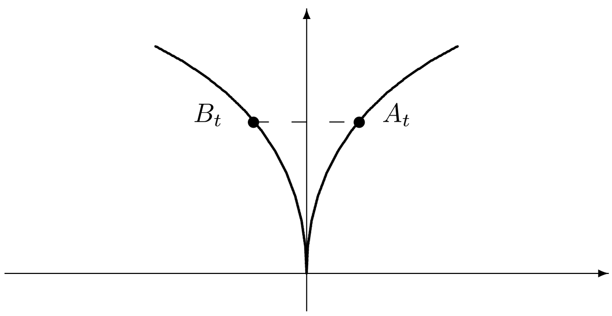
$$g : X \to \mathbb{R},\ (x, y) \mapsto g(x, y) = \begin{cases} y & \text{if } x \geq 0, \\ -y & \text{if } x \leq 0. \end{cases}$$

Let us see that $g \in D^\infty(X) \setminus \mathrm{LIP}^\infty(X)$.

Indeed, if $t \neq 0$, it can be checked that $\operatorname{Lip} g(t^3, t^2) \leq 1$. On the other hand, at the origin we have

$$\operatorname{Lip} g(0,0) = \limsup_{(x,y)\to(0,0)} \frac{|g(x,y) - g(0,0)|}{d((x,y),(0,0))} = \limsup_{t\to 0} \frac{t^2}{\sqrt{(t^3)^2 + (t^2)^2}} = 1.$$

Thus, we obtain that $\|\operatorname{Lip} f\|_\infty = 1$ and so $g \in D^\infty(X)$. Take now two symmetric points from the cusp with respect to the y-axis, that is, $A_t = (t^3, t^2)$ and $B_t = (-t^3, t^2)$ for $0 < t < 1$.



In this case, we get $d(A_t, B_t) = 2t^3$ and $|f(A_t) - f(B_t)| = t^2 - (-t^2) = 2t^2$. If t tends to 0, we have

$$\lim_{t\to 0^+} \frac{|f(A_t) - f(B_t)|}{d(A_t, B_t)} = \lim_{t\to 0^+} \frac{2t^2}{2t^3} = \lim_{t\to 0^+} \frac{1}{t} = +\infty.$$

Thus, g is not a Lipschitz function. $\square$

In general, we have that

$$\begin{array}{ccccccc} & & & \rotatebox{45}{$\subsetneq$} & \mathrm{LIP}_{\mathrm{loc}}(X) & \rotatebox{-45}{$\subsetneq$} & \\ \mathrm{LIP}(X) & \subsetneq & \mathrm{LIP}_{\mathrm{loc}}(X) \cap D(X) & & & & \mathcal{C}(X) \\ & & & \rotatebox{-45}{$\subsetneq$} & D(X) & \rotatebox{45}{$\subsetneq$} & \end{array}$$

where $\mathrm{LIP}_{\mathrm{loc}}(X)$ denotes the space of locally Lipschitz functions. Recall that in Example II.1.5 we have constructed a function $f \in \mathrm{LIP}_{\mathrm{loc}}(X) \cap D(X) \setminus \mathrm{LIP}(X)$. In addition, there is no inclusion relation between $\mathrm{LIP}_{\mathrm{loc}}(X)$ and $D(X)$. Indeed, consider for instance the metric space $X = \bigcup_{i=1}^{\infty} B_i \subset \mathbb{R}$ with the Euclidean distance where $B_i = B(i, 1/3)$ denotes the open ball centered at i and radius 1/3. One can check that the function $f(x) = ix$ if $x \in B_i$ is locally Lipschitz whereas $f \notin D(X)$ because $\|\operatorname{Lip} f\|_{\infty} = \infty$. On the other hand, the function g in Example II.1.6 belongs to $D(X) \setminus \mathrm{LIP}_{\mathrm{loc}}(X)$.

II.2 A Banach space structure for pointwise Lipschitz functions

In this section we search for sufficient conditions to have a converse of Corollary II.1.4. We begin introducing a kind of metric spaces which will play a central role throughout this section. In addition, for such spaces, we will endow the space of functions $D^{\infty}(X)$ and $D(X)$ with a Banach structure.

Definition II.2.1. Let (X, d) be a metric space. We say that X is *locally radially quasiconvex* if for each $x \in X$, there exist a neighborhood U^x and a constant $K_x > 0$ such that for each $y \in U^x$ there exists a rectifiable curve γ in U^x connecting x and y such that $\ell(\gamma) \le K_x d(x, y)$.

Note that the spaces introduced in the Examples II.1.5 and II.1.6 are locally radially quasiconvex. Observe that there exist locally radially quasiconvex spaces which are not locally quasiconvex (see Example II.1.6).

There also exist locally rectifiably connected spaces which are not locally radially quasiconvex. Recall that X is *locally rectifiably connected* if for each $x \in X$, there exists a neighborhood U^x such that for each $y \in U^x$ there exists a rectifiable curve γ in U^x connecting x and y.

Example II.2.2. Let (X, d_0) the metric space we have defined in II.1.5. For each $n \ge 1$, we define the metric space (X_n, d_n) where $X_n = [0, +\infty) \times \{n\}$ with the distance

$$d_n((x, n), (y, n)) = d_0(x + n, y + n).$$

Next, we consider $X = \bigcup_{n \ge 1} X_n$ and identify the points $(0, n)$ for each $n \ge 1$ with a single point O and define the distance between two points $p, q \in X$ in the following way:

- If $p, q \in X_n$ for any n, then $d(p, q) = d_n(p, q)$.
- If $p \in X_n$ and $y \in X_m$ with $n \ne m$, then $d(p, q) = d_n(p, O) + d_m(O, q)$.

Let us check that (X, d) is a locally rectifiably connected space. Let $r > 0$ and consider the ball $B(O, r) = \{p \in X : \ d(O, p) < r\}$. It is clear that any point in the ball can be connected to O through a rectifiable curve. As we have seen in II.1.5, for each $n \geq 1$ and each integer $a \geq 1$ we have that

$$d(O,(a,n)) = d_0(n, a+n) \leq \frac{2}{n}a.$$

For each n, we choose an integer $a_n \geq 1$ such that $\frac{2}{n}a_n < r$ and $\frac{2}{n}(a_n+1) \geq r$. In particular, $(a_n,n) \in B(O,r)$ and $\lim_{n\to\infty} a_n = \infty$. On the other hand, the shortest rectifiable curve connecting O with (a_n,n) has length a_n since the metric coincides locally with the Euclidean one. We deduce that (X,d) is not a locally radially quasiconvex space. $\square$

Next, we endow the space $D^\infty(X)$ with the following norm:

$$\|f\|_{D^\infty} = \max\{\|f\|_\infty, \|\operatorname{Lip} f\|_\infty\},$$

for each $f \in D^\infty(X)$.

It is clear that $\|\cdot\|_{D^\infty} \leq \|\cdot\|_{\mathrm{LIP}^\infty}$. However, in general, both norms are not equivalent. For example, if we consider the space $X = [0,1]\cup[2,3]$ and the function $f(x) = 0$ if $x \in [0,1]$ and $f(x) = 1$ if $x \in [2,3]$ it is clear that f is a 1-Lipschitz function but $\operatorname{Lip} f(x) = 0$ for every $x \in X$.

Theorem II.2.3. *Let (X,d) be a locally radially quasiconvex metric space. Then, $(D^\infty(X), \|\cdot\|_{D^\infty})$ is a Banach space.*

Proof. Let $\{f_n\}_n$ be a Cauchy sequence in $(D^\infty(X), \|\cdot\|_{D^\infty})$. Since $\{f_n\}_n$ is uniformly Cauchy, there exists $f \in \mathcal{C}(X)$ such that $f_n \to f$ with the norm $\|\cdot\|_\infty$.

Let us first prove that $\|\operatorname{Lip}(f_n - f)\|_\infty \to 0$. Let $\varepsilon > 0$ be given. There exists $n_0 \in \mathbb{N}$ such that $\|f_n - f_m\|_{D^\infty} \leq \varepsilon$ for all $n,m \geq n_0$. Now fix $x \in X$. Since (X,d) is locally radially quasiconvex, there exist a neighborhood U^x of x and a constant $K_x > 0$ such that that for each $y \in U^x$ there exists a rectifiable curve γ which connects x and y such that $\ell(\gamma) \leq K_x\, d(x,y)$. Now there exists $n_x \geq n_0$ such that, for every $j,k \geq n_x$

$$\|\operatorname{Lip}(f_j - f_k)\|_\infty \leq \frac{\varepsilon}{K_x}.$$

By Lemma II.1.3, we find that for every $y \in U^x$ and every $j,k \geq n_x$:

$$|f_j(x) - f_k(x) - (f_j(y) - f_k(y))| \leq \|\operatorname{Lip}(f_j - f_k)\|_\infty\, K_x\, d(x,y) \leq \varepsilon\, d(x,y).$$

Choosing $j = n_x$ and letting $k \to \infty$ we obtain that, for every $y \in U^x$:

$$|(f_{n_x} - f)(x) - (f_{n_x} - f)(y))| \leq \varepsilon\, d(x,y).$$

Now let $n \geq n_0$ and consider the function $f_n - f_{n_x} \in D^\infty(X)$. Since $\mathrm{Lip}(f_n - f_{n_x}) \leq \|f_n - f_{n_x}\|_{D^\infty} \leq \varepsilon$, there exists a neighborhood W_n^x of x contained in U^x such that, for every $y \in W_n^x$,

$$|(f_n - f_{n_x})(x) - (f_n - f_{n_x})(y))| \leq 2\,\varepsilon\, d(x,y).$$

Thus for every $y \in W_n^x$ we have that

$$|(f_n - f)(x) - (f_n - f)(y))| \leq |(f_n - f_{n_x})(x) - (f_n - f_{n_x})(y)|$$
$$+|(f_{n_x} - f)(x) - (f_{n_x} - f)(y))| \leq 3\,\varepsilon\, d(x,y).$$

Therefore $\mathrm{Lip}(f_n - f)(x) \leq 3\,\varepsilon$ for every $n \geq n_0$ and every $x \in X$. In this way we obtain that $\|\mathrm{Lip}(f_n - f)\|_\infty \leq 3\,\varepsilon$ for every $n \geq n_0$.

Now since $(f_n - f) \in D^\infty(X)$ for $n \geq n_0$, we deduce that also $f \in D^\infty(X)$. Furthermore, we have that $f_n \to f$ for $\|\cdot\|_{D^\infty}$.

□

Let us see however that in general $(D^\infty(X), \|\cdot\|_{D^\infty})$ is not a Banach space.

Example II.2.4. Consider the connected metric space $X = X_0 \cup \bigcup_{n=1}^\infty X_n \cup G \subset \mathbb{R}^2$ with the metric induced by the Euclidean one, where $X_0 = \{0\} \times [0, +\infty)$, $X_n = \{\frac{1}{n}\} \times [0, n]$, $n \in \mathbb{N}$ and $G = \{(x, \frac{1}{x}) : 0 < x \leq 1\}$. For each $n \in \mathbb{N}$ consider the sequence of functions $f_n : X \to [0,1]$ given by

$$f_n\Big(\frac{1}{k}, y\Big) = \begin{cases} \frac{k-y}{k\sqrt{k}} & \text{if } 1 \leq k \leq n \\ 0 & \text{if } k > n, \end{cases}$$

and $f_n(x,y) = 0$ if $x \neq \frac{1}{k}$ $\forall k \in \mathbb{N}$. Observe that $f_n(\frac{1}{k}, 0) = \frac{1}{\sqrt{k}}$ and $f_n(\frac{1}{k}, k) = 0$ if $1 \leq k \leq n$. Since $\mathrm{Lip}\, f_n(\frac{1}{k}, y) = \frac{1}{k\sqrt{k}}$ and $\mathrm{Lip}\, f_n(x,y) = 0$ if $x \neq \frac{1}{k}$ $\forall k \in \mathbb{N}$, we have that $f_n \in D^\infty(X)$ for each $n \geq 1$. In addition, if $1 < n < m$,

$$\|f_n - f_m\|_\infty = \frac{1}{\sqrt{n+1}} \qquad \text{and} \qquad \|\mathrm{Lip}(f_n - f_m)\|_\infty = \frac{1}{(n+1)\sqrt{n+1}}.$$

Thus, we deduce that $\{f_n\}_n$ is a Cauchy sequence in $(D^\infty(X), \|\cdot\|_{D^\infty})$. However, if $f_n \to f$ in D^∞ then $f_n \to f$ pointwise. Therefore $f_m(\frac{1}{n}, 0) = \frac{1}{\sqrt{n}}$ for each $m \geq n$ and so $f(\frac{1}{n}, 0) = \frac{1}{\sqrt{n}}$ and $f(0,0) = 0$. Thus, we obtain that

$$\mathrm{Lip}(f)(0,0) \geq \lim_{n\to\infty} \frac{|f(\frac{1}{n}, 0) - f(0,0)|}{d((\frac{1}{n}, 0), (0,0))} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{\frac{1}{n}} = +\infty,$$

and so $f \notin D^\infty(X)$. This means that $(D^\infty(X), \|\cdot\|_{D^\infty})$ is not a Banach space.

Remark II.2.5. The previous example can be modified to obtain a path-connected metric space X such that $(D^\infty(X), \|\cdot\|_{D^\infty})$ is not complete. For example, one can connect X_0 to $\{1\} \times [0,1]$ by a curve that does not intersect any of the X_n, $n \geq 2$.

Theorem II.2.6. *Let (X,d) be a connected locally radially quasiconvex metric space and let $x_0 \in X$. If we consider on $D(X)$ the norm $\|f\|_D = \max\{|f(x_0)|, \|\operatorname{Lip}(f)\|_\infty\}$, then $(D(X), \|\cdot\|_D)$ is a Banach space.*

Proof. Let $x \in X$. By hypothesis, for each $y \in X$, there exists a neighborhood U^y such that for each $z \in U^y$, there exists a rectifiable curve in U^y connecting z and y. Since X is connected, there exists a finite sequence of points $y_1, \ldots, y_m$ such that $U^{y_k} \cap U^{y_{k+1}} \neq \emptyset$ for $k = 1, \ldots, m-1$, $x \in U^{y_1}$ and $x_0 \in U^{y_m}$. Now, for each $k = 1 \ldots m$, choose a point $z_k \in U^{y_k} \cap U^{y_{k+1}}$. To simplify notation we write $z_0 = x_0$ and $z_{n+1} = x$. For each $k = 1 \ldots m$, we choose a rectifiable curve γ_k which connects z_k with z_{k+1}. Taking $\gamma = \gamma_0 \cup \ldots \cup \gamma_m$ we obtain a rectifiable curve γ which connects x_0 and x.

Let us see now that $(D(X), \|\cdot\|_D)$ is a Banach space. Indeed, let $\{f_n\}_n$ be a Cauchy sequence. We consider the case on which $f_n(x_0) = 0$ for each $n \geq 1$. The general case can be done in a similar way. By combining the previous argument with Lemma II.1.3, we obtain that for $n, m \geq 1$ and for each $x \in X$

$$|f_n(x) - f_m(x)| \leq \|\operatorname{Lip}(f_n - f_m)\|_\infty \ell(\gamma),$$

where γ is a rectifiable curve connecting x and x_0. Since $\{f_n\}_n$ is a Cauchy sequence with respect to the seminorm $\|\operatorname{Lip}(\cdot)\|_\infty$, the sequence $\{f_n(x)\}_n$ is a Cauchy sequence for each $x \in X$, and therefore, it converges to a point $y = f(x)$. Then, in particular, $\{f_n\}_n$ converges pointwise to a function $f : X \to \mathbb{R}$.

Next, one finds using the same strategy as in Theorem II.2.3 (where we have just used the pointwise convergence) that a Cauchy sequence $\{f_n\}_n \subset D(X)$ such that $f_n(x_0) = 0$ for each $n \geq 1$, converges in $(D(X), \|\cdot\|_D)$ to a function $f \in D(X)$. $\square$

Remark II.2.7. It is well known that $\mathrm{LIP}^\infty(X)$ is a Banach space. As shown before, $D^\infty(X)$ is not a Banach space in general. However, if $\mathrm{LIP}^\infty(X) = D^\infty(X)$ with comparable energy seminorms, then $D^\infty(X)$ is also a Banach space.

We are now prepared to state a partial converse of Corollary II.1.4.

Corollary II.2.8. *Let (X,d) be a complete locally compact connected metric space. Then $\mathrm{LIP}^\infty(X) = D^\infty(X)$ with comparable energy seminorms if and only if X is a quasiconvex space.*

Proof. The fact that if X is a quasiconvex space then $\mathrm{LIP}^\infty(X) = D^\infty(X)$ with comparable energy seminorms is Corollary II.1.4. On the other hand, for each $x \in X$ and $\varepsilon > 0$ we define the ε-*distance* from x to z to be

$$\rho_{x,\varepsilon}(z) := \inf \sum_{i=0}^{N-1} d(z_i, z_{i+1}), \tag{II.2}$$

where the infimum is taken over all finite ε-chains $(z_i)_{i=0}^N$. For positive integers N we set $\rho_{x,\varepsilon,N} = \min\{N, \rho_{x,\varepsilon}\}$. Since X is connected we see that $\rho_{x,\varepsilon,N}$ is finite-valued everywhere and $|\rho_{x,\varepsilon,N}(z) - \rho_{x,\varepsilon,N}(w)| \leq d(z,w)$ when $d(z,w) < \varepsilon$; thus for all $w \in X$ we have $\operatorname{Lip} \rho_{x,\varepsilon,N}(w) \leq 1$. Hence $\rho_{x,\varepsilon,N}$ belongs to $D^\infty(X)$. Because $\mathrm{LIP}^\infty(X) = D^\infty(X)$ with comparable energy seminorms there is a constant $C > 0$ such that $\mathrm{LIP}(\rho_{x,\varepsilon,N}) \leq C$ with C independent of x, ε, N. It follows that for all $y \in X$ and all $\varepsilon > 0$,

$$|\rho_{x,\varepsilon,N}(y)| = |\rho_{x,\varepsilon,N}(y) - \rho_{x,\varepsilon,N}(x)| \leq \mathrm{LIP}(\rho_{x,\varepsilon,N}) d(x,y) \leq C d(x,y). \tag{II.3}$$

Using a standard limiting argument, which involves Arzela-Ascoli's Theorem and inequality (II.3), we can construct a 1-Lipschitz rectifiable curve connecting x and y with length at most $Cd(x,y)$. Since x and y were arbitrary this completes the proof. For further details about the construction of the curve we refer the reader to [79, Theorem 3.1]. $\square$

We finish the section studying the problem of characterizing when the function space of pointwise Lipschitz function admits a Banach linearization. Linearization is a useful tool for studying function spaces, since it enables the application of linear functional analysis to problems concerning nonlinear functions.

It is well known that the space $\mathrm{LIP}^\infty(X)$ admits always a predual, and more generally admits a *Banach linearization over* X. By this we mean a pair (Z, δ), where Z is a predual of $\mathrm{LIP}^\infty(X)$ and $\delta : X \to Z$ is a continuous mapping "linearizing" the functions of $\mathrm{LIP}^\infty(X)$ in the sense that it verifies:

(1) For every z^* in the dual space Z^*, we have that $z^* \circ \delta \in \mathrm{LIP}^\infty(X)$.

(2) For every $f \in \mathrm{LIP}^\infty(X)$ there exists a unique $z_f^* \in Z^*$ such that $f = z_f^* \circ \delta$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\delta"'] & \mathbb{R} \\ Z \arrow[ur, dashed, "z_f^*"'] & \end{tikzcd}$$

We refer to [64] for further information about this topic. A Banach linearization of $\mathrm{LIP}^\infty(X)$ is obtained by considering the mapping $\delta : X \to \mathrm{LIP}^\infty(X)^*$ defined by $\delta(x) = \delta_x$, the evaluation at the point x. It can be shown that the closed subspace Z of $\mathrm{LIP}^\infty(X)^*$ spanned by the evaluations $\{\delta_x : x \in X\}$ is a predual of $\mathrm{LIP}^\infty(X)$, and furthermore conditions (1) and (2) are fulfilled. For a proof of this fact see for example [62] and [66]. Now it is natural to ask whether $D^\infty(X)$, in the case that it is Banach, admits a Banach linearization or at least a predual. We will see that this depends on the geometry of X. We first need an auxiliary lemma.

Lemma II.2.9. *Let (X, d) be a locally radially quasiconvex metric space. Then the mapping $\delta : X \to D^\infty(X)^*$ is continuous, where $\delta(x) = \delta_x$ is the evaluation at the point x.*

Proof. Let $x \in X$. We know that there exist a neighborhood U^x of x and a constant $K_x > 0$ such that for each $y \in U^x$ there exists a rectifiable curve γ which connects x and y such that $\ell(\gamma) \leq K_x\, d(x, y)$. Now let $\{x_n\}_n$ be a sequence in X convergent to x. There exists $n_0 \in \mathbb{N}$ such that $x_n \in U^x$ for all $n \geq n_0$. By Lemma II.1.3, for every $f \in D^\infty(X)$ and every $n \geq n_0$ we have that

$$|f(x) - f(x_n)| \leq \|f\|_{D^\infty} K_x\, d(x, x_n).$$

Therefore

$$\|\delta(x) - \delta(x_n)\| = \sup\{|f(x) - f(x_n)| \,:\, \|f\|_{D^\infty} \leq 1\} \leq K_x\, d(x, x_n),$$

from which $\|\delta(x) - \delta(x_n)\| \to 0$ as $n \to \infty$. □

Now we obtain the following characterization. Here by an *equivalent ball* we mean the closed unit ball for an equivalent norm on the space.

Corollary II.2.10. *Let (X, d) be a locally radially quasiconvex metric space. The following conditions are equivalent:*

(a) *$D^\infty(X)$ admits a Banach linearization.*

(b) *$D^\infty(X)$ admits an equivalent ball which is pointwise-closed on X.*

Proof. Every function in $D^\infty(X)$ is bounded on X. Thus an equivalent ball in $D^\infty(X)$ is pointwise-compact if, and only if, it is pointwise-closed. Then the result follows from Lemma II.2.9 and Theorem 2.2 of [64]. □

Using this we can give a sufficient condition for the existence of linearization. We say that a metric space (X, d) is *uniformly locally radially quasiconvex* if there exists a uniform constant C such that each point $x \in X$ has a neighborhood U^x such that for each $y \in U^x$ there exists a rectifiable curve γ which connects x and y such that $\ell(\gamma) \leq C\, d(x, y)$. This condition is satisfied, for example, for every compact locally radially quasiconvex space.

Corollary II.2.11. *If (X,d) is a uniformly locally radially quasiconvex metric space, $D^\infty(X)$ admits a Banach linearization.*

Proof. Let $C>0$ be a constant satisfying the requirements of the definition of uniform local radial quasiconvexity. If $\{f_\alpha\}_\alpha$ is a net in the closed unit ball $\overline{B}$ of $D^\infty(X)$, which is pointwise convergent on X to some function f, then each point $x\in X$ has a neighborhood U^x such that for every $y\in U^x$ and every α we have:

$$|f_\alpha(x)-f_\alpha(y)|\leq C\,d(x,y).$$

Then for every $y\in U^x$ we also have that $|f(x)-f(y)|\leq C\,d(x,y)$. Thus $\operatorname{Lip} f(x)\leq C$ for every $x\in X$. Therefore, if we consider the pointwise closure $\hat{B}$ of $\overline{B}$, we obtain that $\hat{B}$ is a pointwise-closed equivalent ball in $D^\infty(X)$, and the result follows from Corollary II.2.10 (see also Corollary 2.7 in [64]). □

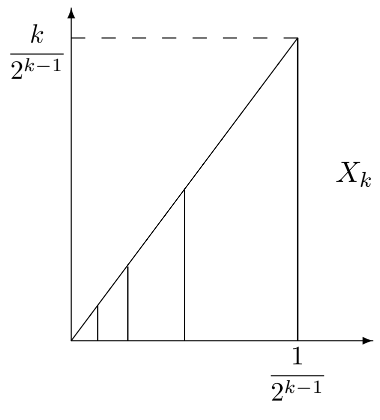
The following example shows that there exists metric spaces (X,d) for which $D^\infty(X)$ is a dual space but it does not admit a Banach linearization over X.

Example II.2.12. For each $k\geq 1$, let us consider the set

$$X_k=\bigcup_{m=k-1}^{\infty} S_m^k\cup\left\{(x,kx):\,0\leq x\leq\frac{1}{2^{k-1}}\right\},$$

where

$$S_m^k=\left\{\frac{1}{2^m}\right\}\times\left[0,\frac{k}{2^m}\right].$$



We define the metric space (X,d) where X is the disjoint union of the family $\{X_k : k\in\mathbb{N}\}$, that is,

$$X=\bigsqcup_k X_k$$

and d is the restriction to X_k of the Euclidean metric of $\mathbb{R}^2$ for each $k \geq 1$ and $d(p,q) = 1$ if $p \in X_k$ and $q \in X_{k'}$ for $k \neq k'$. The elements of the disjoint union are ordered pairs $(p,k) = p_k$, where $p \in X_k$. Here k serves as an auxiliary index that indicates which X_k the element p comes from. The space (X,d) is locally radially quasiconvex. Indeed, outside the origin, each X_k is locally quasiconvex. On the other hand, for each $n \in \mathbb{N}$, the point $(\frac{1}{2^n}, 0)_k$ is connected to the origin $(0,0)_k$ by a curve γ whose length is

$$\ell(\gamma) = \frac{k + \sqrt{1+k^2}}{2^n} = (k + \sqrt{1+k^2})d\big((0,0)_k, (\frac{1}{2^n}, 0)_k\big) \leq 3kd\big((0,0)_k, (\frac{1}{2^n}, 0)_k\big),$$

for each $k \geq 1$. From this computation we deduce that each X_k is $3k$-locally radially quasiconvex. Moreover, a modification of the previous argument shows that actually X_k is $3k$-quasiconvex. Therefore (X,d) is locally radially quasiconvex but it is not uniformly locally radially quasiconvex.

Let us define the sequence of functions

$$f_n\Big(\Big(\frac{1}{2^n}, y\Big)_k\Big) = \frac{k}{2^n} - y \quad \text{for } k = 1 \cdots n,$$

and $f_n(x,y) = 0$ otherwise. It is clear that $f_n \in B_{D^\infty(X)}$ and that $f_n \to f$ pointwise, where the limit function f satisfies that $f((0,0)_k) = 0$ and $f((\frac{1}{2^k}, 0)_k) = k/2^k$ for each $k \geq 1$. Thus, $\operatorname{Lip} f((0,0)_k) \geq k$ and so $\|\operatorname{Lip} f\|_\infty = \infty$. Therefore $B_{D^\infty(X)}$ does not admit any equivalent ball pointwise-closed on X and so by Corollary II.2.10, $D^\infty(X)$ does not admit a Banach linearization over X. However $D^\infty(X)$ is a dual space. Indeed, for each $k \in \mathbb{N}$, using the fact that X_k is $3k$-quasiconvex we have by Corollary II.1.4 that $D^\infty(X_k) = \mathrm{LIP}^\infty(X_k)$ and therefore

$$D^\infty(X_k) = \mathrm{LIP}^\infty(X_k) = Z_k^*,$$

where Z_k is a predual for $\mathrm{LIP}^\infty(X_k)$. In particular

$$D^\infty(X) = (\bigoplus_{k=1}^{\infty} D^\infty(X_k))_{\ell_\infty} = (\bigoplus_{k=1}^{\infty} Z_k^*)_{\ell_\infty} = [(\bigoplus_{k=1}^{\infty} Z_k)_{\ell_1}]^*,$$

where

$$(\bigoplus_{k=1}^{\infty} E_k)_{\ell_\infty} = \{(x_k)_{k\in\mathbb{N}} : x_k \in E_k \text{ and } \|(x_k)\|_{\ell_\infty} = \sup_k \|x_k\|_{E_k} < \infty\}.$$

II.3 A Banach-Stone Theorem for pointwise Lipschitz functions

There exist many results in the literature relating the topological structure of a topological space X with the algebraic or topological-algebraic structures of certain function spaces defined on it. The classical Banach-Stone theorem asserts that for a compact space X, the linear metric structure of $\mathcal{C}(X)$ endowed with the sup-norm determines the topology of X. Results along this line for spaces of Lipschitz functions have been recently obtained in [47, 48]. In this section we prove two versions of the Banach-Stone theorem for the function spaces $D^\infty(X)$ and $D(X)$ respectively, where X is a locally radially quasiconvex space. Since in general $D(X)$ has not an algebra structure we will consider on it its natural unital vector lattice structure. On the other hand, on $D^\infty(X)$ we will consider both, its algebra and its unital vector lattice structures.

It can be easily checked that we have a Leibniz's rule in this context, that is, if $f, g \in D^\infty(X)$, then $\|\operatorname{Lip}(f \cdot g)\|_\infty \leq \|\operatorname{Lip} f\|_\infty \|g\|_\infty + \|\operatorname{Lip} g\|_\infty \|f\|_\infty$. In this way, we can always endow the space $D^\infty(X)$ with a natural algebra structure. Note that $D^\infty(X)$ is *uniformly separating* in the sense that for every pair of subsets A and B of X with $d(A, B) > 0$, there exists some $f \in D^\infty(X)$ such that $\overline{f(A)} \cap \overline{f(B)} = \emptyset$. In our case, if A and B are subsets of X with $d(A, B) = \alpha > 0$, then the function $f = \inf\{d(\cdot, A), \alpha\} \in \mathrm{LIP}^\infty(X) \subset D^\infty(X)$ satisfies that $f = 0$ on A and $f = \alpha$ on B. In addition, we can endow either $D^\infty(X)$ or $D(X)$ with a natural unital vector lattice structure.

We denote by $\mathcal{H}(D^\infty(X))$ the set of all nonzero algebra homomorphisms $\varphi : D^\infty(X) \to \mathbb{R}$, that is, the set of all nonzero multiplicative linear functionals on $D^\infty(X)$. Note that in particular every algebra homomorphism $\varphi \in \mathcal{H}(D^\infty(X))$ is positive, that is, $\varphi(f) \geq 0$ when $f \geq 0$. Indeed, if f and $1/f$ are in $D^\infty(X)$, then $\varphi(f \cdot (1/f)) = 1$ implies that $\varphi(f) \neq 0$ and $\varphi(1/f) = 1/\varphi(f)$. Thus, if we assume that φ is not positive, then there exists $f \geq 0$ with $\varphi(f) < 0$. The function $g = f - \varphi(f) \geq -\varphi(f) > 0$, satisfies $g \in D^\infty(X)$, $1/g \in D^\infty(X)$ and $\varphi(g) = 0$ which is a contradiction.

Now, we endow $\mathcal{H}(D^\infty(X))$ with the topology of pointwise convergence (that is, considered as a topological subspace of $\mathbb{R}^{D^\infty(X)}$ with the product topology). This construction is standard (see for instance [63]), but we give some details for completeness. It is easy to check that $\mathcal{H}(D^\infty(X))$ is closed in $\mathbb{R}^{D^\infty(X)}$ and therefore is a compact space. In addition, since $D^\infty(X)$ separates points and closed sets, X can be embedded as a topological subspace of $\mathcal{H}(D^\infty(X))$ identifying each $x \in X$ with the point evaluation homomorphism δ_x given by $\delta_x(f) = f(x)$, for every $f \in D^\infty(X)$. We are going to see that X is dense in $\mathcal{H}(D^\infty(X))$. Indeed, given $\varphi \in \mathcal{H}(D^\infty(X))$, $f_1, \ldots, f_n \in D^\infty(X)$, and $\varepsilon > 0$, there exists some $x \in X$ such that $|\delta_x(f_i) - \varphi(f_i)| < \varepsilon$, for $i = 1, \ldots, n$. Otherwise, the function $g = \sum_{i=1}^n |f_i - \varphi(f_i)| \in D^\infty(X)$ would satisfy $g \geq \varepsilon$ and $\varphi(g) = 0$, and this is impossible since φ is positive. It follows that $\mathcal{H}(D^\infty(X))$ is a compactification of X. Moreover, every $f \in D^\infty(X)$ admits a continuous extension to $\mathcal{H}(D^\infty(X))$, namely by defining $\widehat{f}(\varphi) = \varphi(f)$ for all $\varphi \in \mathcal{H}(D^\infty(X))$.

Lemma II.3.1. *Let (X, d) be a metric space and $\varphi \in \mathcal{H}(D^\infty(X))$. Then, $\varphi : D^\infty(X) \to \mathbb{R}$ is a continuous map.*

Proof. Let $f \in D^\infty(X)$. We know that it admits a continuous extension $\widehat{f} : \mathcal{H}(D^\infty(X)) \to$

$\mathbb{R}$ so that $\widehat{f}(\varphi) = \varphi(f)$. Thus, since X is dense in $\mathcal{H}(D^\infty(X))$,

$$|\varphi(f)| = |\widehat{f}(\varphi)| \leq \sup_{\eta \in \mathcal{H}(D^\infty(X))} |\widehat{f}(\eta)| = \sup_{x \in X} |f(x)| \leq \|f\|_{D^\infty}$$

and we are done. □

We next give some results which will give rise to a Banach-Stone theorem for $D^\infty(X)$.

Lemma II.3.2. *Let (X, d_X) and (Y, d_Y) be locally radially quasiconvex metric spaces. Then, every unital algebra homomorphism $T : D^\infty(X) \to D^\infty(Y)$ is continuous for the respective D^∞-norms.*

Proof. First recall that, by Theorem II.2.3, $D^\infty(X)$ and $D^\infty(Y)$ are Banach spaces. Thus, in order to prove the continuity of the linear map T, we can apply the Closed Graph Theorem. Then it is enough to check that given a sequence $\{f_n\}_n \subset D^\infty(X)$ with $\|f_n - f\|_{D^\infty}$ convergent to zero and $g \in D^\infty(X)$ such that $\|T(f_n) - g\|_{D^\infty}$ also convergent to zero, then $T(f) = g$. Indeed, let $y \in Y$, and let $\delta_y \in \mathcal{H}(D^\infty(Y))$ be the homomorphism given by the evaluation at y, that is, $\delta_y(h) = h(y)$. By Lemma II.3.1, we have that $\delta_y \circ T \in \mathcal{H}(D^\infty(X))$ is continuous and so

$$T(f_n)(y) = (\delta_y \circ T)(f_n) \to (\delta_y \circ T)(f) = T(f)(y)$$

when $n \to \infty$.

On the other hand, since convergence in D^∞-norm implies pointwise convergence, then $T(f_n)(y)$ converges to $g(y)$. That is, $T(f)(y) = g(y)$, for each $y \in Y$. Hence, $T(f) = g$ as wanted. □

The concept of real-valued pointwise Lipschitz function can be generalized in a natural way when the target space is a metric space.

Definition II.3.3. Let (X, d_X) and (Y, d_Y) be metric spaces. Given a function $f : X \to Y$ we define

$$\operatorname{Lip} f(x) = \limsup_{\substack{y \to x \\ y \neq x}} \frac{d_Y(f(x), f(y))}{d_X(x, y)},$$

for each nonisolated $x \in X$. If x is an isolated point we define $\operatorname{Lip} f(x) = 0$. We consider the following space of functions

$$D(X, Y) = \{f : X \longrightarrow Y \ : \|\operatorname{Lip} f\|_\infty < +\infty\}.$$

As we have seen in Lemma II.1.2 we may observe that if $f \in D(X, Y)$ then f is continuous.

As a consequence of Lemma II.3.2, we obtain the following result concerning the composition of pointwise Lipschitz functions.

Proposition II.3.4. *Let (X, d_X) and (Y, d_Y) be locally radially quasiconvex metric spaces and let $h : X \to Y$. Suppose that $f \circ h \in D^\infty(X)$ for each $f \in D^\infty(Y)$. Then $h \in D(X, Y)$.*

Proof. We begin by checking that h is a continuous map, that is, $h^{-1}(C)$ is closed in X for each closed subset C in Y. Let C be a closed subset of Y and suppose that there exists some $y_0 \in Y \backslash C$. Take $f = \inf\{d_Y(\cdot, C), d_Y(y_0, C)\} \in D^\infty(Y)$. Let us observe that $f^{-1}(\{0\}) = C$. Thus, since $f \circ h$ is continuous, we obtain that $h^{-1}(C) = h^{-1}(f^{-1}(\{0\})) = (f \circ h)^{-1}(\{0\})$ is closed in X.

By Lemma II.3.2, the homomorphism $T : D^\infty(Y) \to D^\infty(X)$ given by $T(f) = f \circ h$ is continuous, and so, there exists $K > 0$ such that $\|f \circ h\|_{D^\infty(X)} \leq K\|f\|_{D^\infty(Y)}$, for each $f \in D^\infty(Y)$.

Note that if $x_0 \in X$ is an isolated point, we have that $\operatorname{Lip} h(x_0) = 0$. Now, let $x_0 \in X$ be a nonisolated point. Let $f_{x_0} = \min\{d_Y(\cdot, h(x_0)), 1\} \in \mathrm{LIP}^\infty(Y) \subset D^\infty(Y)$ which has $\mathrm{LIP}(f_{x_0}) = 1$. In particular, $\|\operatorname{Lip}(f_{x_0})\|_\infty \leq 1$ and $\|f_{x_0}\|_\infty \leq 1$, and so $\|f_{x_0}\|_{D^\infty(Y)} \leq 1$. In addition, we have that

$$\begin{aligned}\operatorname{Lip}(f_{x_0} \circ h)(x_0) &= \limsup_{\substack{y \to x_0 \\ y \neq x_0}} \frac{|f_{x_0} \circ h(y) - f_{x_0} \circ h(x_0)|}{d_X(x_0, y)} = \limsup_{\substack{y \to x_0 \\ y \neq x_0}} \frac{|f_{x_0} \circ h(y)|}{d_X(x_0, y)} \\ &= \limsup_{\substack{y \to x_0 \\ y \neq x_0}} \frac{\min\{d_Y(h(y), h(x_0)), 1\}}{d_X(x_0, y)} = \operatorname{Lip} h(x_0),\end{aligned}$$

where the last equality above holds because, as we have checked before, the map h is continuous. Thus, we obtain that

$$\begin{aligned}\operatorname{Lip} h(x_0) &= \operatorname{Lip}(f_{x_0} \circ h)(x_0) \leq \|\operatorname{Lip}(f_{x_0} \circ h)\|_\infty \leq \|f_{x_0} \circ h\|_{D^\infty(X)} \\ &\leq K\|f_{x_0}\|_{D^\infty(Y)} \leq K.\end{aligned}$$

We conclude that $\|\operatorname{Lip} h\|_\infty \leq K$, and the proof is now complete. □

Finally, we need the following useful Lemma, which shows that the points in X can be topologically distinguished into $\mathcal{H}(D^\infty(X))$. It is essentially known (see for instance [46]) but we give a proof for completeness.

Lemma II.3.5. *Let (X, d) be a complete metric space and let $\varphi \in \mathcal{H}(D^\infty(X))$. Then φ has a countable neighborhood basis in $\mathcal{H}(D^\infty(X))$ if, and only if, $\varphi \in X$.*

Proof. Suppose first that $\varphi \in \mathcal{H}(D^\infty(X)) \backslash X$ has a countable neighborhood basis. Since X is dense in $\mathcal{H}(D^\infty(X))$, there exists a sequence (x_n) in X converging to φ. The completeness of X implies that (x_n) has no Cauchy sequence in (X, d), and therefore there exist $\varepsilon > 0$ and a subsequence (x_{n_k}) such that $d(x_{n_k}, x_{n_j}) \geq \varepsilon$ for $k \neq j$. Now, the sets $A = \{x_{n_k} : k \text{ even }\}$ and $B = \{x_{n_k} : k \text{ odd }\}$ satisfy $d(A, B) \geq \varepsilon$, and since $D^\infty(X)$ is uniformly separating, there is a function $f \in D^\infty(X)$ with $\overline{f(A)} \cap \overline{f(B)} = \emptyset$. But this is a contradiction since f extends continuously to $\mathcal{H}(D^\infty(X))$ and φ is in the closure of both A and B.

Conversely, if $\varphi \in X$, let B_n be the open ball in X with center φ and radius $1/n$. For each n, there exists an open subset V_n of $\mathcal{H}(D^\infty(X))$ such that $B_n = V_n \cap X$. Since X is dense in $\mathcal{H}(D^\infty(X))$, it is easily seen that the closure $\mathrm{cl}_{\mathcal{H}}(B_n)$ of B_n in $\mathcal{H}(D^\infty(X))$ coincides with the closure of V_n. On the other hand, since $\mathcal{H}(D^\infty(X))$ is compact, every point has a neighborhood basis consisting of closed sets. Using this, it is not difficult to see that the family $\{\mathrm{cl}_{\mathcal{H}}(B_n)\}_n$ is a countable neighborhood basis of φ in $\mathcal{H}(D^\infty(X))$. □

Now, we are in a position to show that the algebra structure of $D^\infty(X)$ determines the pointwise Lipschitz structure of a complete locally radially quasiconvex metric space. We say that two metric spaces X and Y are *pointwise Lipschitz homeomorphic* if there exists a bijection $h : X \to Y$ such that $h \in D(X, Y)$ and $h^{-1} \in D(Y, X)$.

Theorem II.3.6. (Banach-Stone type) *Let (X, d_X) and (Y, d_Y) be complete locally radially quasiconvex metric spaces. The following are equivalent:*

(a) *X is pointwise Lipschitz homeomorphic to Y.*

(b) *$D^\infty(X)$ is isomorphic to $D^\infty(Y)$ as unital algebras.*

(c) *$D^\infty(X)$ is isomorphic to $D^\infty(Y)$ as unital vector lattices.*

Proof. (a) $\Longrightarrow$ (b) If $h : X \to Y$ is a pointwise Lipschitz homeomorphism, then it is easy to check the map $T : D^\infty(Y) \to D^\infty(X)$, $f \mapsto T(f) = f \circ h$, is an isomorphism of unital algebras.

(b) $\Longrightarrow$ (a) Let $T : D^\infty(X) \to D^\infty(Y)$ be an isomorphism of unital algebras. We define $h : \mathcal{H}(D^\infty(Y)) \to \mathcal{H}(D^\infty(X))$, $\varphi \mapsto h(\varphi) = \varphi \circ T$. Let us see first that h is an homeomorphism. To reach that aim, it is enough to prove that h is bijective, closed and continuous. Since T is an isomorphism, $h^{-1}(\psi) = \psi \circ T^{-1}$ exists for every $\psi \in \mathcal{H}(D^\infty(X))$, and so h is bijective. In addition, once we check that h is continuous we will also have that h is closed because $\mathcal{H}(D^\infty(Y))$ is compact and $\mathcal{H}(D^\infty(X))$ is a Hausdorff space. Now consider the following diagram:

$$\begin{tikzcd}
Y \arrow[r, hook] \arrow[d, "T(f)"'] & \mathcal{H}(D^\infty(Y)) \arrow[r, "h"] \arrow[d, "\widehat{T(f)}"'] \arrow[rd, dashed, "\widehat{f}\circ h"'] & \mathcal{H}(D^\infty(X)) \arrow[d, "\widehat{f}"] & X \arrow[l, hook'] \arrow[d, "f"] \\
\mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R}
\end{tikzcd}$$

Here, $\widehat{f}$ (respectively $\widehat{T(f)}$) denotes the continuous extension of f (respectively $T(f)$) to $\mathcal{H}(D^\infty(X))$. Thus, h is continuous if and only if $\widehat{f} \circ h$ is continuous for all $f \in D^\infty(X)$.

Hence, it is enough to prove that $\widehat{f} \circ h = \widehat{T(f)}$. Since X is dense in $\mathcal{H}(D^\infty(X))$, it is suffices to check that

$$\widehat{T(f)}(\delta_x) = \widehat{f} \circ h(\delta_x),$$

where δ_x denotes the evaluation homomorphism for each $x \in X$. It is clear that,

$$\widehat{f} \circ h(\delta_x) = (h \circ \delta_x)(f) = (\delta_x \circ T)(f) = \delta_{T(f)(x)} = \delta_x(Tf) = \widehat{T(f)}(\delta_x),$$

and so h is continuous.

By Lemma II.3.5 we have that a point $\varphi \in \mathcal{H}(D^\infty(X))$ has a countable neighborhood basis in $\mathcal{H}(D^\infty(X))$ if and only if it corresponds to a point of X. Since the same holds for Y and $\mathcal{H}(D^\infty(Y))$ we conclude that $h(Y) = X$ and by Proposition II.3.4 we have that $h|_Y \in D(Y, X)$. Analogously, $h^{-1}|_X \in D(X, Y)$ and so X and Y are pointwise Lipschitz homeomorphic.

To prove (b) $\iff$ (c) We use that $D^\infty(X)$ is *closed under bounded inversion* which means that if $f \in D^\infty(X)$ and $f \geq 1$, then $1/f \in D^\infty(X)$. Indeed, if $f \in D^\infty(X)$ and $f \geq 1$, given $\varepsilon > 0$ there exists $r > 0$ such that

$$\frac{|f(x) - f(y)|}{d(x,y)} \leq \sup_{\substack{d(x,y) \leq r \\ y \neq x}} \frac{|f(x) - f(y)|}{d(x,y)} \leq M + \varepsilon \qquad \forall y \in B(x,r). \tag{II.4}$$

Thus, given $x \in X$,

$$\Big| \frac{1}{f(y)} - \frac{1}{f(x)} \Big| = \frac{|f(x) - f(y)|}{|f(x)f(y)|} \leq d(x,y)(M + \varepsilon) \qquad \forall y \in B(x,r), \tag{II.5}$$

where inequality (II.5) is obtained after applying (II.4) and the fact that $|f(x)f(y)| \geq 1$. Thus, the conclusion follows from Lemma 2.3 in [47]. $\square$

Corollary II.3.7. *Let (X, d_X) and (Y, d_Y) be complete locally radially quasiconvex metric spaces. The following assertions are equivalent:*

(a) *X is pointwise Lipschitz homeomorphic to Y.*

(b) *$D(X)$ is isomorphic to $D(Y)$ as unital vector lattices.*

Proof. (a) $\Longrightarrow$ (b) If $h : X \to Y$ is a pointwise Lipschitz homeomorphism, then it is clear that the map $T : D(Y) \to D(X)$, $f \mapsto T(f) = f \circ h$, is an isomorphism of unital vector lattices.

(b) $\Longrightarrow$ (a) It follows from Theorem II.3.6, since each homomorphism of unital vector lattices $T : D(Y) \to D(X)$ takes bounded functions to bounded functions. Indeed, if $|f| \leq M$ then $|T(f)| = T(|f|) \leq T(M) = M$. $\square$

(II.3.8) Non complete case. If X is a metric space and $\widetilde{X}$ denotes its completion, then both metric spaces have the same uniformly continuous functions. Therefore, $\mathrm{LIP}(X) = \mathrm{LIP}(\widetilde{X})$, and completeness of spaces cannot be avoided in the Lipschitzian case. We are interested in how completeness assumption works for the D-case. It would be useful to analyze if there exists a Banach-Stone theorem for not complete metric spaces.

Example II.3.9. Let (X, d) be the metric space given by

$$X = \{(x, y) \in \mathbb{R}^2 : y^3 = x^2, -1 \leq x \leq 1\} = \{(t^3, t^2), -1 \leq t \leq 1\},$$

where d is the restriction to X of the Euclidean metric of $\mathbb{R}^2$. Let (Y, d') be the metric space given by $Y = X \backslash \{0\}$ and $d' = d|_Y$. Observe that (X, d) is the completion of (Y, d'). The function

$$h : Y \to \mathbb{R},\ (x, y) \mapsto \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0, \end{cases}$$

belongs to $D(Y)$ but h cannot be even continuously extended to X. Thus, $D(Y) \neq D(X)$.

In the following example we construct a metric space X such that $D(X) = D(\widetilde{X})$, where $\widetilde{X}$ denotes the completion of X, and so that X is not homeomorphic to $\widetilde{X}$. This fact illustrates that, a priori, one cannot expect a conclusive result for the non complete case.

Example II.3.10. Let X be a metric space defined as follows:

$$X = \{(t^3, t^2), -1 \leq t \leq 1\} \cup \{(x, 1) \in \mathbb{R}^2 : 1 \leq x < 2\} = A \cup B.$$

Now, we consider the completion of X :

$$\widetilde{X} = \{(t^3, t^2), -1 \leq t \leq 1\} \cup \{(x, 1) \in \mathbb{R}^2 : 1 \leq x \leq 2\} = \widetilde{A} \cup \widetilde{B}.$$

Let $f \in D(X)$. First of all, $D(B) = \mathrm{LIP}(B)$, since B is a quasiconvex space, and so, by McShane's Theorem (see [55, 6.2]), there exists $F \in \mathrm{LIP}(\widetilde{B})$ such that $F|_B = f$. Thus,

$$G(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in A = \widetilde{A} \\ F(x, y) & \text{if } (x, y) \in \widetilde{B}, \end{cases}$$

is a D-extension of f to the completion $\widetilde{X}$. And so $D(X) = D(\widetilde{X})$. However, X is not homeomorphic to $\widetilde{X}$ since $\widetilde{X}$ is compact but X is not.

(II.3.11) Extensions. Lipschitz functions are D-functions and it is important to observe that in every metric space there are plenty of non trivial real-valued Lipschitz functions. McShane's Extension Theorem states that every Lipschitz function $f : A \to \mathbb{R}$ defined on a subset A of a metric space X can be extended to a Lipschitz funtion $f : X \to \mathbb{R}$ (see [55, 6.2]). In this context, it is a natural question if it is possible to extend a function $f \in D^\infty(A)$, where A is a subset of a metric space X, to a function $F \in D^\infty(X)$ satisfying $F|_A = f$. In general it is not possible to find such an extension. Indeed, let $X = B((0,0),3) \subseteq \mathbb{R}^2$ and denote by A the set described in II.1.6. The function g that appears in the example belongs to $D^\infty(A)$ but does not belong to $\mathrm{LIP}^\infty(A)$. If there exists a function $F \in D^\infty(X)$ satisfying $F|_A = f$, we would have, since X is a chainable space, that $F \in D^\infty(X) = \mathrm{LIP}^\infty(X)$. If we restrict F to A we obtain that $F|_A \in \mathrm{LIP}^\infty(A)$, which gives us a contradiction.

II.4 Pointwise Isometries

Next we deal with what we call *pointwise isometries* between metric spaces, related to pointwise Lipschitz functions.

Definition II.4.1. Let (X, d_X) and (Y, d_Y) be metric spaces. We say that X and Y are *pointwise isometric* if there exists a bijection $h : X \to Y$ such that $\|\operatorname{Lip} h\|_\infty = \|\operatorname{Lip} h^{-1}\|_\infty = 1$.

Remark II.4.2. If a bijection $h : X \to Y$ between two metric spaces satisfies that $\operatorname{Lip} h \leq 1$ and $\operatorname{Lip} h^{-1} \leq 1$, then $\operatorname{Lip} h(x) = 1$ for each $x \in X$ and $\operatorname{Lip} h^{-1}(y) = 1$ for each $y \in Y$. Applying the definition of the operator Lip we have that given $\varepsilon > 0$, there exists $r > 0$ such that

$$\frac{d_Y(h(x), h(y))}{d_X(x,y)} \leq (1+\varepsilon) \qquad \forall y \in B(x,r).$$

If we take superior limits when y tends to x in both terms of the inequality we obtain that $\operatorname{Lip} h(x) \leq 1$. Analogously, there exists $s > 0$ such that

$$d_X(h^{-1}(h(x)), h^{-1}(z)) \leq d_Y(h(x), z)(1+\varepsilon) \qquad \forall z \in B(h(x), s).$$

The function h is bijective and so, $h^{-1}(z) = y$ for some $y \in h^{-1}(B(h(x), s))$. Since h is continuous, there exists $t > 0$ such that $B(x,t) \subset h^{-1}(B(h(x), s))$. Thus,

$$\frac{1}{1+\varepsilon} \leq \frac{d_Y(h(x), h(y))}{d_X(x,y)} \qquad \forall y \in B(x,t).$$

If we take superior limits when y tends to x, we obtain that $\operatorname{Lip} h(x) \geq 1$. And so, $\operatorname{Lip} h(x) = 1$ for each $x \in X$. Analogously, we have $\operatorname{Lip} h^{-1}(y) = 1$ for each $y \in Y$.

Lemma II.4.3. Let (X, d_X) and (Y, d_Y) be complete locally radially quasiconvex metric spaces. There exists an isomorphism of vector lattices $T : D^\infty(Y) \to D^\infty(X)$ which is an isometry for the $\|\cdot\|_{D^\infty}$ norms (that is, $\|T\| = \|T^{-1}\| = 1$) if and only if X and Y are pointwise isometric.

Proof. We deduce from the proofs of Proposition II.3.4 and Theorem II.3.6 that if there exists an isomorphism of vector lattices $T : D^\infty(Y) \to D^\infty(X)$ which is an isometry for the $\|\cdot\|_{D^\infty}$ norms (that is, $\|T\| = \|T^{-1}\| = 1$), then by the previous remark we obtain that X and Y are pointwise isometric.

Conversely, let X and Y be pointwise isometric. Then, there exists a bijection $h : X \to Y$ such that $\|\operatorname{Lip} h\|_\infty = \|\operatorname{Lip} h^{-1}\|_\infty = 1$. We have that $T : D^\infty(Y) \to D^\infty(X)$, defined by $f \to T(f) = f \circ h$ is an isomorphism of vector lattices. To finish, let us compute the following norm:

$$\|T\| = \sup_{\|f\|_{D^\infty(Y)}=1} \|f \circ h\|_{D^\infty(X)}.$$

Since h is surjective and $\|f\|_{D^\infty(Y)} = 1$ we have that $\|f \circ h\|_\infty = \|f\|_\infty \leq 1$ and also $\|\operatorname{Lip}(f \circ h)\|_\infty \leq \|\operatorname{Lip} f\|_\infty \cdot \|\operatorname{Lip} h\|_\infty \leq 1$ (chain's rule). Thus $\|f \circ h\|_{D^\infty(X)} \leq 1$ and so $\|T\| \leq 1$. In fact, if we take the constant function $f = 1$, we obtain that $\|f \circ h\|_{D^\infty(X)} = 1$, and so $\|T\| = 1$. Analogously we obtain that $\|T^{-1}\| = 1$ and so T is an isometry for the $\|\cdot\|_{D^\infty}$ norms. □

It is clear that if two metric spaces are locally isometric, then they are pointwise isometric. The converse is not true, as we can see throughout the following example.

Example II.4.4. Let (X, d_X) be the metric space introduced in Example II.1.6 and let (Y, d_Y) be the metric space defined in the following way. Consider the interval $Y = [-1, 1]$ and let us define a metric on it as follows:

$$d_Y(t,s) = \begin{cases} d_X((t^3,t^2),(s^3,s^2)) & \text{if } t,s \in [-1,0], \\ d_X((t^3,t^2),(s^3,s^2)) & \text{if } t,s \in [0,1], \\ d_X((t^3,t^2),(0,0)) + d_X((0,0),(s^3,s^2)) & \text{if } t \in [-1,0], s \in [0,1]. \end{cases}$$

Let us see that d_Y defines a metric. We just check the triangle inequality. Let $t, s, p \in [-1, 1]$ be such that $t \leq s \leq p$. If we denote by $\alpha_t = (t^3, t^2)$, $\alpha_s = (s^3, s^2)$, $\alpha_p = (p^3, p^2)$ and $\alpha_0 = (0, 0)$ we have that

$$\begin{aligned} d_Y(t,p) =& d_X(\alpha_t,\alpha_p) \leq d_Y(t,s) + d_Y(s,p) \\ =& \begin{cases} d_X(\alpha_t,\alpha_s) + d_X(\alpha_s,\alpha_0) + d_X(\alpha_0,\alpha_p) & \text{if } t,s \in [-1,0] \text{ and } p \in [0,1], \\ d_X(\alpha_t,\alpha_0) + d_X(\alpha_0,\alpha_s) + d_X(\alpha_s,\alpha_p) & \text{if } t \in [-1,0] \text{ and } s,p \in [0,1], \\ d_X(\alpha_t,\alpha_s) + d_X(\alpha_s,\alpha_p) & \text{if } t,s,p \in [-1,0] \text{ or } t,s,p \in [0,1], \end{cases} \end{aligned}$$

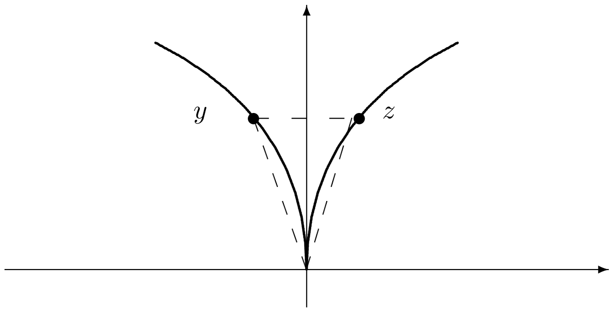
where all the inequalities hold since d_X satisfies the triangle inequality. We define

$$h : X \to Y,\ (t^3, t^2) \to t.$$

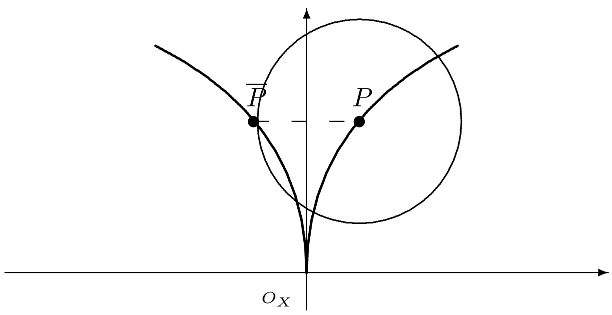
Let us observe that $\| \operatorname{Lip} h\|_\infty = \| \operatorname{Lip} h^{-1}\|_\infty = 1$ and so X and Y are pointwise isometric. However, at the origin $(0,0)$, for each $r > 0$ we have that

$$d_X(z,y) \neq d_Y(h(z),h(y)) \quad \forall z,y \in B((0,0),r).$$

Thus, h is a pointwise homomorphism but not a local isometry.



Let us check that in fact there is no local isometry $f : X \longrightarrow Y$. Suppose by the contrary that there exists a local isometry $f : X \longrightarrow Y$. Let O_X denote the origin seen as a point of X and let O_Y denote the origin as a point of Y. Let U_{O_X} be a neighborhood of O_X such that $f|_{U_{O_X}}$ is an isometry. We may assume that U_{O_X} is the intersection with X of an open ball centered at O_X. Let $P = (x,y) \in U_{O_X}$ and let $\overline{P} = (-x,y)$ the symmetric point of P with respect to the axis $\{x = 0\}$. Consider the set $\{Q \in U_{O_X} : d_X(Q,P) = d_X(P,\overline{P})\}$ which has at least three elements. This fact can be seen in the picture below.



Consider also the set $\{x \in f(U_{O_X}) : d_Y(f(P),x) = d_Y(P,\overline{P})\}$, which contains, as one checks straightforwardly, just two elements, a contradiction since $f|_{U_{O_X}}$ is an isometry.

Chapter III

Sobolev spaces on metric measure spaces: $N^{1,\infty}$

Our aim in this chapter is to compare the function spaces $D^\infty(X)$ and $\mathrm{LIP}^\infty(X)$ studied in Chapter II with certain Sobolev spaces on metric-measure spaces.

If we have a measure on the metric space, we can deal with many more problems. In this line, there are several generalizations of classical Sobolev spaces to the setting of arbitrary metric measure spaces. Hajłasz was the first who introduced Sobolev type spaces in this context [52]. He defined the spaces $M^{1,p}(X)$ for $1 \le p \le \infty$ in connection with maximal operators. Shanmugalingam in [96] introduced, using the notion of upper gradient (and more generally weak upper gradient) the Newtonian spaces $N^{1,p}(X)$ for $1 \le p < \infty$. There are another interesting notions of Sobolev spaces to the context of metric measure spaces; see for example [49],[45],[67],[78]. However, under suitable conditions, all the approaches turn to be equivalent [99, 98]. The overview article [53] by Hajłasz presents further generalizations of Sobolev spaces in metric measure spaces. It should be pointed out, that if the space supports a p-Poincaré inequality, $1 < p < \infty$ (see definition III.3.2), all the approaches to Sobolev spaces described in [53] are also equivalent (see [73, Theorem 1.0.6]).

In Section III.1 we will review the notions of Hajłasz-Sobolev space $M^{1,p}(X)$ for $1 \le p \le \infty$ and Newtonian spaces $N^{1,p}(X)$ for $1 \le p < \infty$. In Section III.2 we generalize the Newtonian space for the case $p = \infty$. To be able to introduce the space $N^{1,\infty}(X)$ we will need first the corresponding definitions and main properties of ∞-*modulus of a family of curves* (see Definition III.2.1) and ∞-*weak upper gradients* (see Definition III.2.7). We further define the ∞-*capacity* (see Definition III.2.13), a useful ingredient when proving that $N^{1,\infty}(X)$ is a Banach space. This will be done in Theorem III.2.17. In Section III.3 we will compare the function spaces $D^\infty(X)$ and $\mathrm{LIP}^\infty(X)$ with such Sobolev space, $N^{1,\infty}(X)$. One of the main results of Chapter III is Theorem III.3.3. It enables us to construct, for the class of doubling metric spaces with a p-Poincaré inequality, quasiconvex curves which avoid zero measure sets. This is a technical result which allows us to compare the function spaces $D^\infty(X)$ and $\mathrm{LIP}^\infty(X)$ with the Sobolev space, $N^{1,\infty}(X)$. In particular, for metric spaces with a doubling measure and a p-Poincaré inequality we prove in Corollary III.3.5 the equality of all the mentioned spaces. Furthermore, if we just require a uniform local p-Poincaré inequality we obtain in Theorem III.3.7 that $M^{1,\infty}(X) \subseteq D^\infty(X) = N^{1,\infty}(X)$. Moreover, we will see throughout some examples (III.3.8) that there exist metric spaces X for which $M^{1,\infty}(X) \subsetneq D^\infty(X) = N^{1,\infty}(X)$.

Along this chapter, we always assume that (X, d, μ) is a metric measure space, where μ is a Borel regular measure.

III.1 Sobolev spaces on metric measure spaces

There are several possible extensions of the classical theory of Sobolev spaces to the setting of metric spaces equipped with a Borel measure. Following [4] and [53] we record the definition of $M^{1,p}$ spaces:

(III.1.1) Hajłasz-Sobolev space. For $1 \leq p \leq \infty$ the space $\widetilde{M}^{1,p}(X, d, \mu)$ is defined as the set of all functions $f \in L^p(X)$ for which there exists a function $0 \leq g \in L^p(X)$ such that

$$|f(x) - f(y)| \leq d(x, y)(g(x) + g(y)) \qquad \mu\text{-a.e.} \tag{III.1}$$

As usual, we get the space $M^{1,p}$ after identifying any two functions $f_1, f_2 \in \widetilde{M}^{1,p}(X, d, \mu)$ such that $f_1 = f_2$ μ-a.e. The space $M^{1,p}(X, d, \mu)$ is equipped with the norm

$$\|f\|_{M^{1,p}} = \|f\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over all functions $0 \leq g \in L^p(X)$ that satisfy the requirement (III.1).

In particular, if $p = \infty$ it can be shown that $M^{1,\infty}(X, d, \mu)$ coincides with $\mathrm{LIP}^\infty(X)$ provided that $\mu(B) > 0$ for every open ball $B \subset X$ (see remark 5.1.4 in [4]). In addition, we also have that $1/2\|\cdot\|_{\mathrm{LIP}^\infty} \leq \|\cdot\|_{M^{1,\infty}} \leq \|\cdot\|_{\mathrm{LIP}^\infty}$. In this case we obtain that $M^{1,\infty}(X) = \mathrm{LIP}^\infty(X) \subseteq D^\infty(X)$.

(III.1.2) Newtonian space. Another interesting generalization of Sobolev spaces to general metric spaces are the so-called Newtonian spaces, introduced by Shanmungalingam [96, 95]. Its definition is based on the notion of *upper gradient*. This concept was introduced by Heinonen and Koskela [59] and serves the role of derivatives in a metric space.

A nonnegative Borel function g on X is said to be an *upper gradient* for an extended real-valued function f on X if

$$|f(\gamma(a)) - f(\gamma(b))| \leq \int_\gamma g \tag{III.2}$$

for every rectifiable curve $\gamma : [a, b] \to X$, when both $f(\gamma(a))$ and $f(\gamma(b))$ are finite, and $\int_\gamma g = \infty$ otherwise.

Observe that $g \equiv \infty$ is an upper gradient of every function on X and if there are no rectifiable curves in X then $g \equiv 0$ is an upper gradient of every function on X. If f is Lipschitz, then $g = \mathrm{LIP}(f)$ is an upper gradient for f. Moreover, the local Lipschitz constant $\mathrm{Lip}\, f$ provides us with a smaller upper gradient than the global Lipschitz constant.

On the other hand, each function $f \in W^{1,p}(\mathbb{R}^n)$ has a representative that has a p-integrable upper gradient (see [95]).

We see that the upper gradient plays the role of a derivative in the formula (III.2) which is similar to the one related to the Fundamental Theorem of Calculus. The point is that using upper gradients we may have many of the properties of ordinary Sobolev spaces even though we do not have derivatives of our functions.

If g is an upper gradient of f and $\widetilde{g} = g$ almost everywhere, then it may happen that $\widetilde{g}$ is no longer an upper gradient for f. We do not want our upper gradients to be sensitive to changes on small sets. To avoid this unpleasant situation the notion of *weak upper gradient* is introduced as follows. First we need a way to measure how large a family of curves is. The most important point is if a family of curves is small enough to be ignored. This kind of problem was first approached in [43]. In what follows let $\Upsilon \equiv \Upsilon(X)$ denote the family of all nonconstant rectifiable curves in X. It may happen that $\Upsilon = \emptyset$, but we will be mainly concerned with metric spaces for which the space Υ is large enough. If E is a subset of X then Γ_E^+ is the family of curves γ such that $\mathscr{L}^1(\gamma^{-1}(\gamma \cap E)) > 0$ while Γ_E denotes the family of curves γ such that $\gamma \cap E \neq \emptyset$.

Definition III.1.3. (Modulus of a family of curves) Let $\Gamma \subset \Upsilon$. For $1 \leq p < \infty$ we define the *p-modulus of* Γ by

$$\operatorname{Mod}_p \Gamma = \inf_{\rho} \int_X \rho^p \, d\mu,$$

where the infimum is taken over all nonnegative Borel functions $\rho : X \to [0, \infty]$ such that $\int_\gamma \rho \geq 1$ for all $\gamma \in \Gamma$. If some property holds for all curves $\gamma \in \Upsilon \backslash \Gamma$, such that $\operatorname{Mod}_p \Gamma = 0$, then we say that the property holds for *p-a.e. curve*.

Example III.1.4. (Cusp domains) Fixing $m \in \mathbb{N}$, let $X \subset \mathbb{R}^2$ be the region

$$X := \{(x, y) \in \mathbb{R}^2 \, : \, x \geq 0 \text{ and } 0 \leq y \leq x^m\}$$

be endowed with the Euclidean distance and the 2-dimensional Lebesgue measure $\mathscr{L}^2$.

One can prove that the p-modulus of curves in X passing through the origin is positive if and only if $p > m + 1$. To see this, let ρ be admissible for computing the p-modulus of the family of curves connecting the origin to the vertical line segment $\{1\} \times \mathbb{R}$ inside this domain. For each $0 \leq a \leq 1$ let γ_a be the curve given by $\gamma_a(t) = (t, at^m)$, where t ranges between 0 and 1. Then γ_a is a curve in X connecting the origin to the vertical line segment. Furthermore, the family $\{\gamma_a\}_{0 \leq a \leq 1}$ fibrates $X \cap ([0,1] \times [0,1])$, and since ρ is admissible, we have that

$$\int_{\gamma_a} \rho \, ds \;=\; \int_0^1 \rho \circ \gamma_a(t) \; \sqrt{1 + m^2 a^2 t^{2m-2}} \, dt \;\geq\; 1.$$

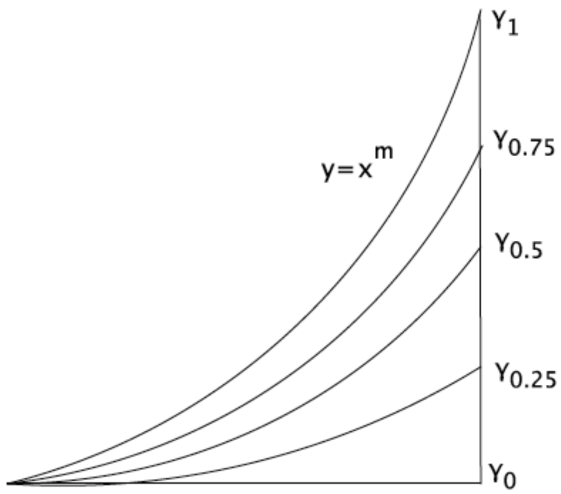


Figure III.1: Fibrating the cusp domain

Thus by Hölder's inequality, with q denoting the Hölder conjugate of p,

$$\begin{aligned} 1 &\leq \int_0^1 \rho \circ \gamma_a(t) t^{m/p}\, t^{-m/p} \sqrt{1+m^2a^2t^{2m-2}}\, dt \\ &\leq \left(\int_0^1 \rho \circ \gamma_a(t)^p t^m \, dt \right)^{1/p} \left(\int_0^1 t^{-mq/p} \left(1+m^2a^2t^{2m-2}\right)^{q/2} \, dt \right)^{1/q}. \end{aligned}$$

Since $1 \leq \sqrt{1+m^2a^2t^{2m-2}} \leq \sqrt{1+m^2}$ for $0 \leq t \leq 1$, the integral

$$C_1 := \int_0^1 t^{-mq/p} \left(1+m^2a^2t^{2m-2}\right)^{q/2} \, dt$$

is finite if and only if $\int_0^1 t^{-mq/p}\, dt$ is finite, and this happens precisely when $p > m+1$. When $p > m+1$, from the above we see that

$$\int_0^1 \rho \circ \gamma_a(t)^p t^m \, dt \geq C_1^{1-p} > 0.$$

It follows that (by setting $\rho(x,y) = 0$ for $x > 1$ without loss of generality),

$$\int_X \rho^p \, d\mathscr{L}^2 = \int_0^1 \int_0^1 \rho \circ \gamma_a(t)^p \, t^m \, dt \, da \geq C_1^{1-p} > 0,$$

and so the p-modulus of the family of all curves passing through the origin, which contains γ_a, $0 \leq a \leq 1$ as a sub-family, is at least $C_1^{1-p} > 0$.

For $1 \le p < m+1$, with the aid of the admissible function $\rho(x,y) = 1/x$, $(x,y) \in X$, we see that the p-modulus of the family of all curves in X passing through the origin is zero. A more careful analysis using the function $\rho(x,y) = (\ln(R/r))^{-1}\, x^{-1}$ for $(x,y) \in X$ which is admissible for computing the p-moduli of curves connecting $\{r\} \times \mathbb{R}$ to $\{R\} \times \mathbb{R}$ in X for $0 < r < R$, and then letting $r \to 0$ also shows that when $p = m+1$ the p-modulus is zero. Observe that the measure $\mathscr{L}^2|_X$ on X is doubling with doubling constant 2^{m+1}.

Definition III.1.5. A nonnegative Borel function g on X is a *p-weak upper gradient* of an extended real-valued function f on X if

$$|f(\gamma(a)) - f(\gamma(b))| \le \int_\gamma g$$

for p-a.e. curve $\gamma \in \Upsilon$.

Let $\widetilde{N}^{1,p}(X,d,\mu)$, where $1 \le p < \infty$, be the class of all L^p integrable Borel functions on X for which there exists a p-weak upper gradient in L^p. For $f \in \widetilde{N}^{1,p}(X,d,\mu)$ we define

$$\|f\|_{\widetilde{N}^{1,p}} = \|f\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over all p-weak upper gradients g of u. Now, we define in $\widetilde{N}^{1,p}$ an equivalence relation by $f_1 \sim f_2$ if and only if $\|f_1 - f_2\|_{\widetilde{N}^{1,p}} = 0$. Then the space $N^{1,p}(X,d,\mu)$ is defined as the quotient $\widetilde{N}^{1,p}(X,d,\mu)/\sim$ and it is equipped with the norm $\|f\|_{N^{1,p}} = \|f\|_{\widetilde{N}^{1,p}}$.

III.2 Newtonian spaces: $N^{1,\infty}(X)$

Next, we consider the case $p = \infty$. We will introduce the corresponding definition of ∞-modulus of a family of rectifiable curves which will be an important ingredient for the definition of the Sobolev space $N^{1,\infty}(X)$.

Definition III.2.1. For $\Gamma \subset \Upsilon$, let $F(\Gamma)$ be the family of all Borel measurable functions $\rho : X \to [0,\infty]$ such that

$$\int_\gamma \rho \ge 1 \quad \text{for all} \quad \gamma \in \Gamma.$$

We define the ∞-*modulus of* Γ by

$$\operatorname{Mod}_\infty \Gamma = \inf_{\rho \in F(\Gamma)} \{\|\rho\|_{L^\infty}\} \in [0,\infty].$$

If some property holds for all curves $\gamma \in \Upsilon \backslash \Gamma$, where $\operatorname{Mod}_\infty \Gamma = 0$, then we say that the property holds for ∞-*a.e. curve*.

Remark III.2.2. It is clear that if a family Γ has ∞-modulus zero, then it has p-modulus zero for all $p < \infty$.

The following Lemma shows that Mod_∞ is an outer measure, as it happens for $1 \leq p < \infty$. See for example Theorem 5.2 in [52].

Lemma III.2.3. *The function* $\mathrm{Mod}_\infty \Gamma : \mathrm{P}(\Upsilon) \to \mathbb{R}$ *is an outer measure on* Υ*, that is,*

(1) $\mathrm{Mod}_\infty(\emptyset) = 0$,

(2) $\Gamma_1 \subset \Gamma_2 \Longrightarrow \mathrm{Mod}_\infty \Gamma_1 \leq \mathrm{Mod}_\infty \Gamma_2$,

(3) $\mathrm{Mod}_\infty(\bigcup_{i=1}^\infty \Gamma_i) \leq \sum_{i=1}^\infty \mathrm{Mod}_\infty \Gamma_i$.

Proof. (1) $\mathrm{Mod}_\infty(\emptyset) = 0$ because $\rho = 0 \in F(\emptyset)$.

(2) If $\Gamma_1 \subset \Gamma_2$, then $F(\Gamma_2) \subset F(\Gamma_1)$ and hence $\mathrm{Mod}_\infty \Gamma_1 \leq \mathrm{Mod}_\infty \Gamma_2$.

(3) We may assume, without loss of generality, that $\sum_{i=1}^\infty \mathrm{Mod}_\infty \Gamma_i < \infty$. For every $j \in \mathbb{N}$ and every $\varepsilon > 0$, let ρ_j be a Borel function such that

$$\int_\gamma \rho_j \geq 1 \text{ for all } \gamma \in \Gamma_j \quad \text{ and } \quad \|\rho_j\|_\infty \leq \mathrm{Mod}_\infty \Gamma_j + \varepsilon/2^j.$$

Let $\rho = \sup_j\{\rho_j\}$ pointwise. We see that ρ satisfies the condition

$$\int_\gamma \rho \geq 1 \quad \text{for all} \quad \gamma \in \bigcup_{j=1}^\infty \Gamma_j.$$

Thus, we get

$$\mathrm{Mod}_\infty \Big(\bigcup_{j=1}^\infty \Gamma_j \Big) \leq \|\rho\|_\infty \leq \sum_{j=1}^\infty \|\rho_j\|_\infty \leq \sum_{j=1}^\infty \mathrm{Mod}_\infty \Gamma_j + \varepsilon,$$

as wanted. $\square$

Remark III.2.4. Notice that, if we have two measures μ and λ defined on X with the same zero measure sets, then the ∞- modulus of Γ is the same, independent of the measure we use to compute it.

Next, we provide a characterization of path families whose ∞-modulus is zero.

Lemma III.2.5. *Let* $\Gamma \subset \Upsilon$*. The following conditions are equivalent:*

(a) $\mathrm{Mod}_\infty \Gamma = 0$.

(b) *There exists a Borel function* $0 \le \rho \in L^\infty(X)$ *such that* $\int_\gamma \rho = +\infty$*, for each* $\gamma \in \Gamma$.

(c) *There exists a Borel function* $0 \le \rho \in L^\infty(X)$ *such that* $\int_\gamma \rho = +\infty$*, for each* $\gamma \in \Gamma$ *and* $\|\rho\|_{L^\infty} = 0$.

Proof. (a) $\Longrightarrow$ (b) If $\mathrm{Mod}_\infty \Gamma = 0$, for each $n \in \mathbb{N}$ there exists $\rho_n \in F(\Gamma)$ such that $\|\rho_n\|_{L^\infty} < 1/2^n$. Let $\rho = \sum_{n\ge 1} \rho_n$. Then $\|\rho\|_{L^\infty} \le \sum_{n=1}^\infty 1/2^n = 1$ and $\int_\gamma \rho = \int_\gamma \sum_{n\ge1} \rho_n = \infty$.

(b) $\Longrightarrow$ (a) On the other hand, let $\rho_n = \rho/n$ for all $n \in \mathbb{N}$. By hypothesis $\int_\gamma \rho_n = \infty$ for all $n \in \mathbb{N}$ and $\gamma \in \Gamma$. Then $\rho_n \in F(\Gamma)$ and $\|\rho\|_{L^\infty}/n \to 0$ as $n \to \infty$. Hence $\mathrm{Mod}_\infty \Gamma = 0$.

(b) $\Longrightarrow$ (c) By hypothesis there exists a Borel measurable function $0 \le \rho \in L^\infty(X)$ such that,

$$\int_\gamma \rho = +\infty \text{ for every } \gamma \in \Gamma.$$

Consider the function

$$h(x) = \begin{cases} \|\rho\|_{L^\infty} & \text{if } \|\rho\|_{L^\infty} \ge \rho(x), \\ \infty & \text{if } \rho(x) > \|\rho\|_{L^\infty}. \end{cases}$$

Notice that $\|\rho\|_{L^\infty} = \|h\|_{L^\infty}$, and since $\int_\gamma \rho = +\infty$ for every $\gamma \in \Gamma$ and $\rho \le h$, we have that $\int_\gamma h = +\infty$ for every $\gamma \in \Gamma$. Now, we define the function $\varrho = h - \|h\|_{L^\infty}$ which has $\|\varrho\|_{L^\infty} = 0$ and

$$\int_\gamma \varrho = \int_\gamma h - \|h\|_{L^\infty} \ell(\gamma) = +\infty \text{ for every } \gamma \in \Gamma.$$

$\square$

Lemma III.2.6. *Let* $E \subset X$*. If* $\mu(E) = 0$*, then* $\mathrm{Mod}_\infty(\Gamma_E^+) = 0$*.*

Proof. By the properties of the measure μ, we know that there exists a Borel set $F \supset E$ such that $\mu(F \setminus E) = 0$ and so $\mu(F) = 0$. Let $g = \infty \cdot \chi_F$. For $\gamma \in \Gamma_E^+$, we have that $\mathscr{L}^1(\gamma^{-1}(\gamma \cap F)) > 0$ and so $\int_\gamma g ds = \int_{\gamma\cap F} g ds = \infty$. Hence, by Lemma III.2.5 $\mathrm{Mod}_\infty(\Gamma_E^+) = 0$. $\square$

Now we are ready to define the notion of ∞-*weak upper gradient* .

Definition III.2.7. A nonnegative Borel function g on X is an ∞-*weak upper gradient* of an extended real-valued function f on X if

$$|f(\gamma(a)) - f(\gamma(b))| \le \int_\gamma g$$

for ∞-a.e. curve $\gamma \in \Upsilon$.

The following lemma shows that ∞-weak upper gradients can be replaced (except for a zero measure set) by upper gradients.

Lemma III.2.8. *Let g be an ∞-weak upper gradient of f. Then there is an upper gradient $\widetilde{g}$ of f such that $\widetilde{g} \geq g$ everywhere, and $\widetilde{g} = g$ μ-a.e.*

Proof. We denote Γ the family of curves for which g is not an ∞-weak upper gradient for f. We know that $\operatorname{Mod}_\infty \Gamma{=}0$. By Lemma III.2.5 there exists a Borel measurable function $0 \leq \rho \in L^\infty(X)$ such that, $\int_\gamma \rho = +\infty$ for every $\gamma \in \Gamma$ and $\|\rho\|_{L^\infty} = 0$. Now, it suffices to take $\widetilde{g} = g + \rho$. □

The next Lemma was first proved for $\mathbb{R}^n$ by Fuglede [43, Theorem 3 (f)].

Lemma III.2.9. *Let $g_i : X \longrightarrow \mathbb{R} \cup \{-\infty + \infty\}$ be a sequence of Borel functions which converge to a Borel function g in $L^\infty(X)$. Then, there exists a subsequence $(g_{i_j})_j$ such that*

$$\int_\gamma |g_{i_j} - g| \longrightarrow 0 \qquad \textit{as} \qquad j \to \infty,$$

for ∞-a.e. curve $\gamma \in \Upsilon$.

Proof. Let us choose a subsequence $(g_{i_j})_j$ such that $\|g_{i_j} - g\|_{L^\infty} < 2^{-j}$ for each j. Let Γ be the family of curves $\gamma \in \Gamma$ such that $\int_\gamma |g_{i_j} - g|$ does not converge to zero as $j \to \infty$. We will show that $\operatorname{Mod}_\infty \Gamma = 0$. Denote by Γ_j the family of curves in Υ for which $\int_\gamma |g_{i_j} - g| > 2^{-j}$. Then, $2^j |g_{i_j} - g| \in F(\Gamma_j)$ and hence $\operatorname{Mod}_\infty(\Gamma_j) \leq \|g_{i_j} - g\|_{L^\infty} < 2^{-j}$. This, and the fact that $\Gamma \subset \bigcup_{j=i}^\infty \Gamma_j$ for every i implies that $\operatorname{Mod}_\infty \Gamma = 0$. □

Lemma III.2.10. *If g is an ∞-weak upper gradient of f and $\widetilde{g}$ is another nonnegative Borel function such that $\widetilde{g} = g$ μ-a.e., then $\widetilde{g}$ is an ∞-weak upper gradient for f.*

Proof. Let $E = \{x \in X : |g - \widetilde{g}| \neq 0\}$ which is a set of zero measure. By Lemma III.2.6, $\operatorname{Mod}_\infty(\Gamma_E^+) = 0$. Therefore $\int_\gamma |g - \widetilde{g}| = 0$ for ∞-a.e curve and so $\widetilde{g}$ is also an ∞-weak upper gradient for f. □

Let $\widetilde{N}^{1,\infty}(X, d, \mu)$, be the class of all Borel functions $f \in L^\infty(X)$ for which there exists an ∞-weak upper gradient in $L^\infty(X)$. For $f \in \widetilde{N}^{1,\infty}(X, d, \mu)$ we define

$$\|f\|_{\widetilde{N}^{1,\infty}} = \|f\|_{L^\infty} + \inf_g \|g\|_{L^\infty},$$

where the infimum is taken over all ∞-weak upper gradients g of f.

Lemma III.2.8 shows that in the definition of $\widetilde{N}^{1,\infty}$ and $\|\cdot\|_{\widetilde{N}^{1,\infty}}$, ∞-weak upper gradients can be replaced by upper gradients.

Definition III.2.11. (Newtonian space for $p=\infty$) We define an equivalence relation in $\widetilde{N}^{1,\infty}$ by $f_1 \sim f_2$ if and only if $\|f_1 - f_2\|_{\widetilde{N}^{1,\infty}} = 0$. Then the space $N^{1,\infty}(X,d,\mu)$ is defined as the quotient $\widetilde{N}^{1,\infty}(X,d,\mu)/\sim$ and it is equipped with the norm

$$\|f\|_{N^{1,\infty}} = \|f\|_{\widetilde{N}^{1,\infty}}.$$

Note that if $f_1 \in \widetilde{N}^{1,\infty}(X)$ and $f_1 = f_2$ μ-a.e., then it is not necessarily true that $f_2 \in \widetilde{N}^{1,\infty}(X)$. Indeed, let $(X=[-1,1], |\cdot|, \mathscr{L}^1)$. Let $f_1 : X \to \mathbb{R}$ be the function $f_1 = 1$ and $f_2 : X \to \mathbb{R}$ given by $f_2 = 1$ if $x \neq 0$ and $f_2(x) = \infty$ if $x = 0$. In this case we have that $f_1 = f_2$ μ-a.e., $f_1 \in \widetilde{N}^{1,\infty}(X)$ but $f_2 \notin \widetilde{N}^{1,\infty}(X)$.

It follows from Lebesgue Differentiation Theorem that, if μ is doubling, then μ-a.e. $x \in X$ is a Lebesgue point of $N^{1,\infty}(X,d,\mu)$.

Lemma III.2.12. *Let $f_1, f_2 \in \widetilde{N}^{1,\infty}(X,d,\mu)$ such that $f_1 = f_2$ μ-a.e. Then $f_1 \sim f_2$, that is, both functions define exactly the same element in $N^{1,\infty}(X,d,\mu)$.*

Proof. For $f = f_1 - f_2$ we have that $f \in \widetilde{N}^{1,\infty}(X)$ and $\|f\|_{L^\infty} = 0$. To prove that $f_1 \sim f_2$ it suffices to show that $f \circ \gamma = 0$ for ∞-a.e. $\gamma \in \Upsilon$ (because then 0 is an ∞-weak upper gradient of f and so $\|f\|_{N^{1,\infty}} = 0$). Let us define the zero-measure set $E = \{x \in X : f(x) \neq 0\}$ and the function $g = \chi_E \cdot \infty \in L^\infty(X)$ for which $\int_\gamma g < \infty$ for ∞-a.e. curve of Υ and so $g \circ \gamma = 0$ μ-a.e. Thus for all $\gamma \in \Upsilon$ such that $\int_\gamma g < \infty$, we have $\mathscr{L}^1(\gamma^{-1}(E)) = 0$; hence $f \circ \gamma = 0$ μ-a.e., that is, $f \circ \gamma = 0$ on a dense subset of the domain of γ. Therefore, if we prove that $f \circ \gamma$ is a continuous function for ∞-a.e. $\gamma \in \Upsilon$, we will have that $f \circ \gamma = 0$ for ∞-a.e. $\gamma \in \Upsilon$. Indeed, we now prove that $f \circ \gamma$ in absolutely continuous for ∞-a.e. $\gamma \in \Upsilon$. Since $f \in \widetilde{N}^{1,\infty}(X)$, by Lemma III.2.8, there exists an upper gradient $0 \leq g \in L^\infty(X)$ of f. Then, for ∞-a.e. $\gamma \in \Upsilon$ we have

$$|f(\gamma(\beta)) - f(\gamma(\alpha))| \leq \int_\alpha^\beta g(\gamma(\tau))\, d\tau < \infty, \qquad \text{for every} \quad [\alpha,\beta] \subset [0, \ell(\gamma)].$$

Due to the absolute continuity of the integral, we obtain that $f \circ \gamma$ is absolutely continuous and so identically 0, as wanted. □

As noticed above, it is not enough that two functions are equal almost everywhere to be considered as equivalent. The *capacity* is a better tool when studying Newtonian spaces. We give here the definition for the case $p = \infty$ following the one given in [95] for the p-finite case.

Definition III.2.13. The ∞-*capacity* of a set $E \subset X$ with respect to the space $N^{1,\infty}(X)$ is defined by

$$\mathrm{Cap}_\infty(E) = \inf_f \|f\|_{N^{1,\infty}(X)},$$

where the infimum is taken over all functions f in $N^{1,\infty}(X)$ such that $f_{|E} \geq 1$.

Before we prove some properties of the ∞-capacity, we will need the following useful result about convergence of sequence of functions in $N^{1,\infty}(X)$. This result is included for the p-finite case in [95, Section 3].

Proposition III.2.14. *Assume that $f_i \in N^{1,\infty}(X)$ and $g_i \in L^\infty(X)$ is an upper gradient of f_i, for each $i = 1, 2, \ldots$. Assume further that there exist $f, g \in L^\infty(X)$ such that*

(1) *The sequence (f_i) converges to f in $L^\infty(X)$, and*

(2) *the sequence (g_i) converges to g in $L^\infty(X)$.*

Then, there exists a function $\widetilde{f} = f$ μ-a.e. such that g is an ∞- weak upper gradient of $\widetilde{f}$, and in particular $\widetilde{f} \in N^{1,\infty}(X)$.

Proof. Let

$$\widetilde{f}(x) = \frac{1}{2}\Big(\limsup_{i\to\infty} f_i(x) + \liminf_{i\to\infty} f_i(x)\Big).$$

Since $f_i \longrightarrow f \in L^\infty(X)$, in particular it converges μ-a.e. Thus, $\widetilde{f} = f$ μ-a.e. and so $\widetilde{f} \in L^\infty(X)$. The function $\widetilde{f}$ is well-defined outside the zero-measure set

$$E = \{x : \limsup_{i\to\infty} |f_i(x)| = \infty\}.$$

Let Γ be the collection of paths $\gamma \in \Upsilon$ such that either $\int_\gamma g ds = \infty$ or

$$\lim_{i\to\infty} \int_\gamma g_i ds \neq \int_\gamma g ds.$$

By Lemma III.2.9 we know that $\mathrm{Mod}_\infty(\Gamma) = 0$. In addition, since $\mu(E) = 0$ we obtain applying Lemma III.2.6 that $\mathrm{Mod}_\infty(\Gamma_E^+) = 0$. For any nonconstant path γ in the family $\Upsilon \setminus (\Gamma \cup \Gamma_E^+)$ we know that there exists a point $y \in |\gamma| \setminus E$. Since g_i is an upper gradient of f_i, we get for all points $x \in |\gamma|$ that

$$|f_i(x)| - |f_i(y)| \leq |f_i(x) - f_i(y)| \leq \int_\gamma g_i ds.$$

Thus,

$$|f_i(x)| \leq |f_i(y)| + \int_\gamma g_i ds.$$

Taking the supremum limit on both sides of the previous inequality and using the fact that $\gamma \notin \Gamma \cup \Gamma_E^+$ we obtain that

$$\lim_{i\to\infty} |f_i(x)| \leq \lim_{i\to\infty} |f_i(y)| + \int_\gamma g ds < \infty,$$

and so $x \notin E$. In particular, we obtain that $\gamma \notin \Gamma_E$, $\Gamma_E \subset \Gamma \cup \Gamma_E^+$, and so $\mathrm{Mod}_\infty(\Gamma_E) = 0$. This fact will be useful in the proof of Lemma III.2.16.

To finish, let $\gamma \in \Upsilon \setminus \Gamma$, and denote the end of points of γ as x and y. Let us notice by the above argument that $x, y \notin E$, and so one has that

$$\begin{aligned}
|\widetilde{f}(x) - \widetilde{f}(y)| &= \frac{1}{2}\Big| \limsup_{i\to\infty} f_i(x) - \liminf_{i\to\infty} f_i(y) + \liminf_{i\to\infty} f_i(x) - \limsup_{i\to\infty} f_i(y)\Big| \\
&\le \frac{1}{2}\Big(\limsup_{i\to\infty} |f_i(x) - f_i(y)| - \liminf_{i\to\infty} |f_i(x) - f_i(y)|\Big) \\
&\le \frac{1}{2}\Big(\limsup_{i\to\infty} \int_\gamma g_i ds + \liminf_{i\to\infty} \int_\gamma g_i ds\Big) \\
&= \int_\gamma g ds.
\end{aligned}$$

Therefore, g is an ∞-weak upper gradient of $\widetilde{f}$, and so $\widetilde{f} \in N^{1,\infty}(X)$. $\square$

It is easy to see that the set function

$$E \mapsto \mathrm{Cap}_\infty(E),$$

is monotone, $\mathrm{Cap}_\infty(E_1) \le \mathrm{Cap}_\infty(E_2)$ if $E_1 \subset E_2$ and it assigns the value zero to the empty set. The next lemma shows that the ∞-capacity is in addition countably subadditive. All together proves that the ∞-capacity is an outer measure.

Lemma III.2.15. *Let $E_1, E_2, ...$ be arbitrary subsets of X. Then,*

$$\mathrm{Cap}_\infty\Big(\bigcup_{i=1}^\infty E_i\Big) \le \sum_{i=1}^\infty \mathrm{Cap}_\infty(E_i).$$

Proof. First, observe that if $\sum_{i=1}^\infty \mathrm{Cap}_\infty(E_i) = +\infty$, there is nothing to prove. Thus, we may assume that $\sum_{i=1}^\infty \mathrm{Cap}_\infty(E_i) < +\infty$. Let $\varepsilon > 0$. Choose $v_i \in N^{1,\infty}(X)$ with $v_{i|E_i} \ge 1$ and upper gradients h_i of v_i such that

$$\|v_i\|_{L^\infty} + \|h_i\|_{L^\infty} \le \mathrm{Cap}_\infty(E_i) + \frac{\varepsilon}{2^i}.$$

Let

$$f_n = \sum_{i=1}^n |v_i|, \quad \text{and} \quad g_n = \sum_{i=1}^n h_i,$$

where g_n is an upper gradient for f_n. Since $\sum_{i=1}^\infty \mathrm{Cap}_\infty(E_i) < +\infty$, we have that $\sum_{i=1}^\infty \|h_i\|_\infty$ and $\sum_{i=1}^\infty \|v_i\|_\infty$ are bounded by $\sum_{i=1}^\infty \mathrm{Cap}_\infty(E_i) + \varepsilon < +\infty$, so both quantities are finite. This, together with the fact that $\{f_n(x)\}_n$ is a monotone increasing sequence for each $x \in X$ implies that

$$\|f_n - f_m\|_{L^\infty(X)} \leq \sum_{i=m+1}^{n} \|v_i\|_{L^\infty} \longrightarrow 0 \quad \text{as} \quad m \to \infty.$$

Therefore we have that $\{f_n\}_n$ is a Cauchy sequence for the L^∞-norm, and so it converges to $f = \sum_{i=1}^\infty |v_i|$ in the L^∞-norm. Analogously, $\{g_n\}_n$ converges to $g = \sum_{i=1}^\infty h_i$ in the L^∞-norm. We can apply now Proposition III.2.14 to get a function $\widetilde{f} = f$ μ-a.e. such that g is an ∞- weak upper gradient of $\widetilde{f}$, and so $\widetilde{f} \in N^{1,\infty}(X)$. Moreover, $f \geq 1$ in $\bigcup_{i=1}^\infty E_i$ and so, it is admissible for computing $\mathrm{Cap}_\infty\Big(\bigcup_{i=1}^\infty E_i\Big)$. Therefore,

$$\mathrm{Cap}_\infty\Big(\bigcup_{i=1}^\infty E_i\Big) \leq \|\widetilde{f}\|_{N^{1,\infty}} \leq \sum_{i=1}^\infty (\|v_i\|_{L^\infty} + \|h_i\|_{L^\infty}) \leq \sum_{i=1}^\infty \mathrm{Cap}_\infty(E_i) + \varepsilon.$$

The claim follows by letting $\varepsilon \to 0$. □

A corollary of the following lemma is that zero ∞-capacity sets are removable for functions in $N^{1,\infty}(X)$.

Lemma III.2.16. *Let $F \subset X$. If $\mathrm{Cap}_\infty(F) = 0$, then $\mathrm{Mod}_\infty(\Gamma_F) = 0$.*

Proof. Let $\varepsilon > 0$. For each positive integer i we can choose functions $v_i \in N^{1,\infty}(X)$ with $v_{i|F} \geq 1$ and upper gradients h_i of v_i such that

$$\|v_i\|_{L^\infty} + \|h_i\|_{L^\infty} \leq \frac{\varepsilon}{2^i}.$$

Let

$$f_n = \sum_{i=1}^n |v_i|, \quad \text{and} \quad g_n = \sum_{i=1}^n h_i,$$

where g_n is an upper gradient for f_n. As in the proof of Lemma III.2.15 we get a function $\widetilde{f} = \sum_{i=1}^\infty |v_i|$ μ-a.e. such that $\sum_{i=1}^\infty h_i$ is an ∞-weak upper gradient of $\widetilde{f}$, and so $\widetilde{f} \in N^{1,\infty}(X)$. Following the construction in Proposition III.2.14, outside a set E such that $\mathrm{Mod}_\infty(\Gamma_E) = 0$ one can write

$$\widetilde{f}(x) = \lim_{i\to\infty} f_i(x).$$

In addition, $F \subset E$. Indeed, since $v_{i|F} \geq 1$ for each i we get that if $x \in F$, $\widetilde{f}(x) = \infty$ and so $x \in E$. In particular we have that $\Gamma_F \subset \Gamma_E$ and so $\mathrm{Mod}_\infty(\Gamma_F) \leq \mathrm{Mod}_\infty(\Gamma_E) = 0$ and the lemma follows. □

We are now ready to prove that $N^{1,\infty}(X)$ is a Banach space. We essentially follow the proof given in [95, Theorem 3.7] for the p-finite case.

Theorem III.2.17. *$N^{1,\infty}(X)$ is a Banach space.*

Proof. Let $\{f_i\}_i$ be a Cauchy sequence in the $N^{1,\infty}$-norm. By passing to a further subsequence if necessary, it can be assumed that

$$\|f_{j+1}(x) - f_j(x)\|_{N^{1,\infty}} < 2^{-2j},$$

and that $\|g_{j+1,j}\|_{L^\infty} < 2^{-j}$, where $g_{i,j}$ is an upper gradient of $f_i - f_j$.

Our first aim is to construct a candidate to be the limit function of the sequence f_i. We will define that limit function pointwise, and that requires the following auxiliary sets: Let

$$E_j = \{x \in X : |f_{j+1}(x) - f_j(x)| \geq 2^{-j}\},$$

and let

$$F_k = \bigcup_{j=k}^{\infty} E_j \quad \text{and} \quad F = \bigcap_{k=1}^{\infty} F_k.$$

Let us observe that if $x \notin F$, there exists k with

$$|f_{j+1}(x) - f_j(x)| < 2^{-j} \quad \text{for all} \quad j \geq k,$$

and so $\{f_j(x)\}_j$ is a Cauchy sequence in $\mathbb{R}$ which obviously converges. Therefore, we can define $f(x) = \lim_{j\to\infty} f_j(x)$. Let us prove that the set F has ∞-capacity zero. Indeed, the function $2^j\,|f_{j+1}(x) - f_j(x)| \geq 1$ on E_j, so

$$\mathrm{Cap}_\infty(E_j) \leq 2^j\,\|f_{j+1}(x) - f_j(x)\|_{N^{1,\infty}} \leq 2^{-j}.$$

Since ∞-capacity is countably subadditive (see Lemma III.2.15) we get that

$$\mathrm{Cap}_\infty(F_k) \leq \sum_{j=k}^{\infty} \mathrm{Cap}_\infty(E_j) \leq \sum_{j=k}^{\infty} 2^{-j} = 2^{1-k},$$

and thus $\mathrm{Cap}_\infty(F) = 0$.

For $x \in X \setminus F$, the sequence $\{f_j(x)\}_j$ is convergent so we can define

$$f(x) = \lim_{j\to\infty} f_j(x) = f_k(x) + \sum_{j=k}^{\infty} (f_{j+1}(x) - f_j(x)).$$

By Lemma III.2.16 we have that $\mathrm{Mod}_\infty \Gamma_F = 0$. Let $\gamma \in \Upsilon \setminus \Gamma_F$, connecting two points x and y. Then,

$$\begin{aligned}
|(f - f_k)(x) - (f - f_k)(y)| &\leq \sum_{j=k}^{\infty} |(f_{j+1} - f_j)(x) - (f_{j+1} - f_j)(y)| \\
&\leq \sum_{j=k}^{\infty} \int_\gamma g_{j+1,j}\, ds = \int_\gamma \sum_{j=k}^{\infty} g_{j+1,j}\, ds.
\end{aligned}$$

Hence, $\sum_{j=k}^{\infty} g_{j+1,j}$ is an ∞-weak upper gradient of $f - f_k$. Thus,

$$\begin{aligned}
\|f - f_k\|_{N^{1,\infty}} &\leq \|f - f_k\|_{L^\infty} + \sum_{j=k}^{\infty} \|g_{j+1,j}\|_{L^\infty} \\
&\leq \|f - f_k\|_{L^\infty} + \sum_{j=k}^{\infty} 2^{-j} \\
&\leq \|f - f_k\|_{L^\infty} + 2^{-k+1} \to 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Therefore, the subsequence converges in the $N^{1,\infty}$-norm, and we are done. $\square$

III.3 Equality of the Sobolev spaces $M^{1,\infty}(X)$ and $N^{1,\infty}(X)$

In what follows, we will look for conditions under which the Sobolev spaces $M^{1,\infty}(X)$ and $N^{1,\infty}(X)$ coincide. We begin our analysis with the following simple observation.

Lemma III.3.1. *If $f \in D(X)$ then* $\mathrm{Lip}(f)$ *is an upper gradient of f.*

Proof. Let $\gamma : [a, b] \to X$ be a rectifiable curve, parametrized by arc-length, which connects x and y. It can be checked that γ is 1-Lipschitz (see for instance Theorem 3.2 in [52]). The function $f \circ \gamma$ is a pointwise Lipschitz function and by Stepanov Differentiability Theorem (see [100]), it is differentiable μ-a.e. Note that $|(f \circ \gamma)'(t)| \leq \mathrm{Lip}\, f(\gamma(t))$ at every point of $[a, b]$ where $f \circ \gamma$ is differentiable. Now, we deduce that

$$|f(x) - f(y)| \leq \left| \int_a^b (f \circ \gamma)'(t) dt \right| \leq \int_a^b \mathrm{Lip}(f(\gamma(t)))\, dt$$

as wanted. $\square$

If we assume that $\mu(B) > 0$ for every open ball $B \subset X$, it is clear by Lemma III.3.1 that $D^\infty(X) \subset \widetilde{N}^{1,\infty}(X)$ and that the map

$$\begin{array}{ccccc}
\phi & : & D^\infty(X) & \longrightarrow & N^{1,\infty}(X) \\
 & & f & \longrightarrow & [f]
\end{array}$$

is an inclusion. Indeed, if $f_1, f_2 \in D^\infty(X)$ with $[0] = [f_1 - f_2] \in N^{1,\infty}(X)$, we have $f_1 - f_2 = 0$ μ-a.e. Thus $f_1 = f_2$ in a dense subset and since f_1, f_2 are continuous we obtain that $f_1 = f_2$. Therefore we have the following chain of inclusions:

$$\mathrm{LIP}^\infty(X) = M^{1,\infty}(X) \subset D^\infty(X) \subset N^{1,\infty}(X), \tag{III.3}$$

and $\|\cdot\|_{N^{1,\infty}} \leq \|\cdot\|_{D^\infty} \leq \|\cdot\|_{\mathrm{LIP}^\infty} \leq 2\,\|\cdot\|_{M^{1,\infty}}$.

Observe that in general, $D^\infty(X) \neq N^{1,\infty}(X)$. Indeed, the path-connected metric space mentioned in Remark II.2.5 gives an example in which $D^\infty(X)$ is not a Banach space whereas $N^{1,\infty}(X)$ is a Banach space and so $D^\infty(X) \neq N^{1,\infty}(X)$.

Our aim is to obtain the equality of all the spaces in the chain (III.3) above. For that, we need some preliminary terminology and results.

The following Poincaré inequality is now standard in literature on analysis in metric measure spaces. It was introduced in [59].

Definition III.3.2. Let $1 \le p < \infty$. We say that (X, d, μ) supports a *weak p-Poincaré inequality* if there exist constants $C_p > 0$ and $\lambda \ge 1$ such that for every Borel measurable function $f : X \to \mathbb{R} \cup \{\infty, -\infty\}$ and every upper gradient $g : X \to [0, \infty]$ of f, the pair (f, g) satisfies the inequality

$$\fint_{B(x,r)} |f - f_{B(x,r)}|\, d\mu \le C_p\, r \Big(\fint_{B(x,\lambda r)} g^p\, d\mu \Big)^{1/p} \tag{III.4}$$

for each $B(x, r) \subset X$. The word *weak* refers to the possibility that λ may be strictly greater than 1.

Recall here that for arbitrary $A \subset X$ with $0 < \mu(A) < \infty$ we write

$$f_A = \fint_A f = \frac{1}{\mu(A)} \int_A f\, d\mu.$$

The Poincaré inequality creates a link between the measure, the metric and the gradient and it provides a way to pass from the infinitesimal information which gives the gradient to larger scales.

There are several possible definitions for the Poincaré inequality. We can vary the class of functions for which the Poincaré inequality is required to hold, and also replace the upper gradient by other substitutes. For example, instead of considering all measurable functions, it is enough to require inequality (III.4) for compactly supported Lipschitz functions with Lipschitz upper gradients. We can also replace the upper gradient by the local Lipschitz constant. If the space is complete and the measure is doubling (see Definition I.2), most of the reasonable definitions coincide (see [70],[69],[72]). Regarding the size of the balls, if the geometry of the balls is nice enough, for example, if the metric is a length metric, we can assume $\lambda = 1$, but not in general, see [54].

In Chapter IV we will take up again the discussion about Poincaré inequalities.

The proof of the next result is strongly inspired in Proposition 3.2 in [65]. However, we include all the details because of the technical differences, which at certain points become quite subtle.

Theorem III.3.3. *Let X be a complete metric space that supports a doubling Borel measure μ which is nontrivial and finite on balls and suppose that X supports a weak p-Poincaré inequality for some $1 \leq p < \infty$. Let $\rho \in L^{\infty}(X)$ such that $\rho \geq 0$. Then, there exists a set $F \subset X$ of measure 0 and a constant $K > 0$ (depending only on X) such that for all $x, y \in X \setminus F$ there exist a rectifiable curve γ connecting x and y such that $\int_{\gamma} \rho < +\infty$ and $\ell(\gamma) \leq K d(x, y)$.*

Proof. We may assume that $0 \leq \|\rho\|_{L^\infty} \leq 1$. Indeed, in other case, we could take $\widetilde{\rho} = \rho/(1 + \|\rho\|_{L^\infty})$. Let $E = \{x \in X : \ \rho(x) > \|\rho\|_{L^\infty}\}$, which is a set of measure zero.

For each $n \geq 1$ we can choose V_n be an open set such that $E \subset V_n$ and $\mu(V_n) \leq \left(\frac{1}{n2^n}\right)^p$ (see Theorem 1.10 in [85]). In addition, one can choose $V_{n+1} \subset V_n$ for each $n \geq 1$. Note that $E \subseteq \bigcap_{n \geq 1} V_n = E_0$ and $\mu(E_0) = \mu(E) = 0$.

Next, consider the family of functions

$$\rho_n = \|\rho\|_{L^\infty} + \sum_{m \geq n} \chi_{V_m}$$

and the function ρ_0 given by the formula

$$\rho_0(x) = \begin{cases} \|\rho\|_{L^\infty} & \text{if } x \in X \setminus E_0, \\ +\infty & \text{otherwise.} \end{cases}$$

We have the following properties:

1. $\rho_n|_{X \setminus V_n} \equiv \|\rho\|_{L^\infty}$ and $\rho_n|_{E_0} \equiv +\infty$.
2. $\rho \leq \rho_0 \leq \rho_m \leq \rho_n$ if $n \leq m$.
3. ρ_n is lower semicontinuous, since each V_m is open and so, the function $\sum_{m \geq n} \chi_{V_m}$ is lower semicontinuous (see Proposition 7.11 in [42]).
4. $\rho_n - \|\rho\|_{L^\infty} \in L^p(X)$ and $\|\rho_n - \|\rho\|_{L^\infty}\|_{L^p} \leq \frac{1}{n}$. For that, it is enough to prove that $\sum_{m \geq n} \|\chi_{V_m}\|_{L_p} \leq \frac{1}{n}$, which follows from the formula
$$\sum_{m \geq n} \|\chi_{V_m}\|_{L_p} = \sum_{m \geq n} (\mu(V_n))^{1/p} = \sum_{m \geq n} \frac{1}{m2^m} \leq \frac{1}{n} \sum_{m \geq n} \frac{1}{2^m} \leq \frac{1}{n}.$$
5. $\mu(\{M((\rho_n - \|\rho\|_{L^\infty})^p) > 1\}) \leq \frac{C}{n^p}$.
   Indeed, by the Maximal Function Theorem (see (I.5)),
$$\begin{aligned} \mu(\{M((\rho_n - \|\rho\|_{L^\infty})^p) > 1\}) &\leq \frac{C}{1} \int_X |\rho_n - \|\rho\|_{L^\infty}|^p \\ &= C\|\rho_n - \|\rho\|_{L^\infty}\|_{L^p}^p < C\frac{1}{n^p}. \end{aligned}$$

For each $n \geq 1$ consider the set

$$S_n = \{x \in X : \ M((\rho_n - \|\rho\|_{L^\infty})^p)(x) \leq 1\}.$$

We claim that: $S_n \subset S_m$ *if* $n \leq m$ *and* $F = X \setminus \bigcup_{n \geq 1} S_n$ *has measure* 0.

Indeed, if $n \leq m$, we have that $0 \leq \rho_m - \|\rho\|_{L^\infty} \leq \rho_n - \|\rho\|_{L^\infty}$ and so

$$0 \leq (\rho_m - \|\rho\|_{L^\infty})^p \leq (\rho_n - \|\rho\|_{L^\infty})^p;$$

hence $S_n \subset S_m$. On the other hand by (v) above, we have $\mu(X \setminus S_n) \leq \frac{C}{n^p}$. Thus,

$$0 \leq \mu(F) = \mu\Big(X \setminus \bigcup_{n \geq 1} S_n\Big) = \mu\Big(\bigcap_{n \geq 1}(X \setminus S_n)\Big) = \lim_{n \to \infty} \mu(X \setminus S_n) \leq \lim_{n \to \infty} \frac{C}{n^p} = 0.$$

After all this preparatory work, our aim is to prove that there exists a constant $K > 0$ depending only on X such that for all $x, y \in X \setminus F$ there exist a rectifiable curve γ such that $\int_\gamma \rho < +\infty$ and $\ell(\gamma) \leq K d(x,y)$. The constant K will be constructed along the remainder of the proof. In what follows let m_0 be the smallest integer for which $S_{m_0} \neq \emptyset$. Fix $n \geq m_0$ and a point $x_0 \in S_n \subset X \setminus F$. As one can check straightforwardly, it is enough to prove that for each $x \in S_n$ there exists a rectifiable curve γ such that $\int_\gamma \rho < +\infty$ and $\ell(\gamma) \leq K d(x, x_0)$, where the constant K depends only on X and not on x_0 or n.

For our purposes, we define the set Γ_{xy} as the set of all the rectifiable curves connecting x and y. Since a complete metric space X supporting a doubling measure and a weak p-Poincaré inequality is quasiconvex (see Theorem 17.1 in [24]), it is clear that Γ_{xy} is nonempty. We define the function

$$u_n(x) = \inf\Big\{\ell(\gamma) + \int_\gamma \rho_n : \ \gamma \in \Gamma_{x_0 x}\Big\}.$$

Note that $u_n(x_0) = 0$. We will prove that on S_n the function u_n is bounded by a Lipschitz function v_n with a constant K_0 which depends only on X (and not on x_0, n or $\|\rho\|_{L^\infty}$) such that $v_n(x_0) = 0$. Assume this for a moment. We have

$$0 \leq u_n(x) = u_n(x) - u_n(x_0) \leq v_n(x) - v_n(x_0) \leq K_0 d(x, x_0) < (K_0 + 1) d(x, x_0).$$

Thus, there exists a rectifiable curve $\gamma \in \Gamma_{x_0 x}$ such that

$$\ell(\gamma) + \int_\gamma \rho \leq \ell(\gamma) + \int_\gamma \rho_n \leq (K_0 + 1) d(x, x_0).$$

Hence, taking $K = K_0 + 1$, we will have

$$\ell(\gamma) \leq K d(x, x_0) \qquad \text{and} \qquad \int_\gamma \rho < +\infty,$$

as we wanted.

Therefore, consider the functions $u_{n,k} : X \to \mathbb{R}$ given by

$$u_{n,k}(x) = \inf \left\{ \ell(\gamma) + \int_\gamma \rho_{n,k} : \ \gamma \in \Gamma_{x_0 x} \right\}$$

where $\rho_{n,k} = \min\{\rho_n, k\}$ which is a lower semicontinuous function. Let us see that the functions $u_{n,k}$ are Lipschitz for each $k \geq 1$ (and in particular continuous) and that $\rho_{n,k} + 1 \leq \rho_n + 1$ are upper gradients for $u_{n,k}$. Since X is quasiconvex, it follows that $u_{n,k}(x) < +\infty$ for all $x \in X$.

Indeed, let $y, z \in X$, C_q the constant of quasiconvexity for X and $\varepsilon > 0$. We may assume that $u_{n,k}(z) \geq u_{n,k}(y)$. Let $\gamma_y \in \Gamma_{x_0 y}$ be such that

$$u_{n,k}(y) \geq \ell(\gamma_y) + \int_{\gamma_y} \rho_{n,k} - \varepsilon.$$

On the other hand, for each rectifiable curve $\gamma_{yz} \in \Gamma_{yz}$, we have

$$u_{n,k}(z) \leq \ell(\gamma_y \cup \gamma_{yz}) + \int_{\gamma_y \cup \gamma_{yz}} \rho_{n,k},$$

and so

$$|u_{n,k}(z) - u_{n,k}(y)| = u_{n,k}(z) - u_{n,k}(y) \leq \ell(\gamma_{yz}) + \int_{\gamma_{yz}} \rho_{n,k} + \varepsilon = \int_{\gamma_{yz}} (\rho_{n,k} + 1) + \varepsilon.$$

Thus, we obtain that $\rho_{n,k} + 1$ is an upper gradient for $u_{n,k}$. In particular, if $\ell(\gamma_{zy}) \leq C_q d(z, y)$, we deduce that

$$|u_{n,k}(z) - u_{n,k}(y)| \leq (k+1)\ell(\gamma_{zy}) \leq C_q(k+1)d(z, y)$$

and so $u_{n,k}$ is a $C_q(k+1)$-Lipschitz function. Our purpose now is to prove that the restriction to S_n of each function $u_{n,k}$ is a Lipschitz function on S_n with respect to a constant K_0 which depends only on X. Fix $y, z \in S_n$. For each $i \in \mathbb{Z}$, define $B_i = B(z, 2^{-i} d(z, y))$ if $i \geq 1$, $B_0 = B(z, 2d(z, y))$, and $B_i = B(y, 2^i d(z, y))$ if $i \leq -1$. To simplify notation we write $\lambda B(x, r) = B(x, \lambda r)$. In the first inequality of the following estimation we use the fact that, since $u_{n,k}$ is continuous, all points of X are Lebesgue points of $u_{n,k}$. Using the weak p-Poincaré inequality and the doubling condition we get the third inequality. From the Minkowski inequality we deduce the fifth while the last one follows from the definition of S_n:

$$
\begin{aligned}
&|u_{n,k}(z) - u_{n,k}(y)| \le \sum_{i\in\mathbb{Z}} \Big| \fint_{B_i} u_{n,k} d\mu - \fint_{B_{i+1}} u_{n,k} d\mu \Big| \\
&\overset{(*)}{\le} \sum_{i\in\mathbb{Z}} \frac{C_\mu}{\mu(B_i)} \int_{B_i} \Big| u_{n,k} - \fint_{B_i} u_{n,k} d\mu \Big| d\mu \\
&\le C_\mu C_p d(z,y) \sum_{i\in\mathbb{Z}} 2^{-|i|} \Big(\frac{1}{\mu(\lambda B_i)} \int_{\lambda B_i} (\rho_{n,k}+1)^p \Big)^{1/p} \\
&= C_\mu C_p d(z,y) \sum_{i\in\mathbb{Z}} 2^{-|i|} \Big(\frac{1}{\mu(\lambda B_i)} \int_{\lambda B_i} ((\rho_{n,k} - \|\rho\|_{L^\infty}) + \|\rho\|_{L^\infty} + 1)^p \Big)^{1/p} \\
&\le C_\mu C_p d(z,y) \sum_{i\in\mathbb{Z}} 2^{-|i|} \Big(\|\rho\|_{L^\infty} + 1 + \Big(\frac{1}{\mu(\lambda B_i)} \int_{\lambda B_i} (\rho_{n,k} - \|\rho\|_{L^\infty})^p \Big)\Big)^{1/p} \Big) \\
&\le 3\, C_\mu C_p d(z,y) \sum_{i\in\mathbb{Z}} 2^{-|i|} \le K_0 d(z,y)
\end{aligned}
$$

where $K_0 = 9C_\mu C_p$ is a constant that depends only on X. Recall that C_μ is the doubling constant and C_p is the constant which appears in the weak p-Poincaré inequality. Let us see with more detail inequality $(*)$. If $i > 0$, we have that

$$
\begin{aligned}
\Big| \fint_{B_i} u_{n,k} d\mu - \fint_{B_{i+1}} u_{n,k} d\mu \Big| &\le \frac{1}{\mu(B_{i+1})} \Big| \int_{B_{i+1}} \Big(u_{n,k} - \fint_{B_i} u_{n,k}\, d\mu \Big) d\mu \Big| \\
&\le \frac{\mu(B_i)}{\mu(B_i)} \frac{1}{\mu(B_{i+1})} \Big| \int_{B_i} \Big(u_{n,k} - \fint_{B_i} u_{n,k}\, d\mu \Big) d\mu \Big| \\
&\le \frac{C_\mu}{\mu(B_i)} \Big| \int_{B_i} \Big(u_{n,k} - \fint_{B_i} u_{n,k}\, d\mu \Big) d\mu \Big|.
\end{aligned}
$$

We have used that $B_{i+1} \subset B_i$ for $i > 0$ and that μ is a doubling measure and so $\mu(2B_{i+1}) = \mu(B_i) \le C_\mu\, \mu(B_{i+1})$. The cases $i < 0$ and $i = 0$ are similar.

Thus, the restriction of $u_{n,k}$ to S_n is a K_0-Lipschitz function for all $k \ge 1$. Note that $u_{n,k} \le u_{n,k+1}$ and therefore we may define

$$v_n(x) = \sup_k \{u_{n,k}(x)\} = \lim_{k\to\infty} u_{n,k}(x).$$

Whence v_n is a K_0-Lipschitz function on S_n. Since $v_n(x_0) = 0$ and $x_0 \in S_m$ when $m \ge m_0$ we have that $v_n(x) < \infty$ and so, it is enough to check that $u_n(x) \le v_n(x)$ for $x \in S_n$. Now, fix $x \in S_n$. For each $k \ge 1$ there is $\gamma_k \in \varGamma_{x_0 x}$ such that

$$\ell(\gamma_k) + \int_{\gamma_k} \rho_{n,k} \le u_{n,k}(x) + \frac{1}{k} \le K_0 d(x,x_0) + \frac{1}{k}.$$

In particular, $\ell(\gamma_k) \leq K_0 d(x,x_0)+1 := M$ for every $k \geq 1$ and so, by reparametrization, we may assume that γ_k is an M-Lipschitz function and $\gamma_k : [0,1] \to \overline{B(x_0,M)}$ for all $k \geq 1$. Since X is complete and doubling, and therefore closed balls are compact, we are in a position to use the Ascoli-Arzela theorem to obtain a subsequence $\{\gamma_k\}_k$ (which we denote again by $\{\gamma_k\}_k$ to simplify notation) and $\gamma : [0,1] \to X$ such that $\gamma_k \to \gamma$ uniformly. For each k_0, the function $1+\rho_{n,k_0}$ is lower semicontinuous, and therefore by Lemma 2.2 in [65] and the fact that $\{\rho_{n,k}\}_k$ is an increasing sequence of functions, we have

$$\ell(\gamma) + \int_\gamma \rho_{n,k_0} = \int_\gamma (1+\rho_{n,k_0}) \leq \liminf_{k\to\infty} \int_{\gamma_k} (1+\rho_{n,k_0}) \leq \liminf_{k\to\infty} \int_{\gamma_k} (1+\rho_{n,k}).$$

Using the Monotone Convergence Theorem on the left hand side and letting k_0 tend to infinity yields

$$\ell(\gamma) + \int_\gamma \rho_n \leq \liminf_{k\to\infty} \int_{\gamma_k} (1+\rho_{n,k}).$$

Since $\gamma \in \mathit{\Gamma}_{x_0 x}$ we have

$$\begin{aligned} u_n(x) \leq \ell(\gamma) + \int_\gamma \rho_n &\leq \liminf_{k\to\infty} \int_{\gamma_k} (1+\rho_{n,k}) \\ &\leq \liminf_{k\to\infty} \left(u_{n,k}(x) + \frac{1}{k}\right) \leq v_n(x), \end{aligned}$$

and that completes the proof. □

Remark III.3.4. In Theorem III.3.3 we can change the hypothesis of completeness for the space X by local compactness. The proof is analogous to the one of Theorem 1.6 in [65], and we do not include the details.

Corollary III.3.5. *Let X be a complete metric space that supports a doubling Borel measure μ which is nontrivial and finite on balls. If X supports a weak p-Poincaré inequality for $1 \leq p < \infty$, then* $\mathrm{LIP}^\infty(X) = M^{1,\infty}(X) = N^{1,\infty}(X)$ *with equivalent norms.*

Proof. If $f \in N^{1,\infty}(X)$, then there exists an ∞-weak upper gradient $g \in L^\infty(X)$ of f. We denote Γ_1 the family of curves for which g is not an upper gradient for f. Note that $\mathrm{Mod}_\infty \Gamma_1$=0. By Lemma III.2.5 there exists a Borel measurable function $0 \leq \varrho \in L^\infty(X)$ such that, $\int_\gamma \varrho = +\infty$ for every $\gamma \in \Gamma_1$ and $\|\varrho\|_{L^\infty} = 0$. Consider $\rho = g + \varrho \in L^\infty(X)$ which is an upper gradient of f and satisfies that $\|\rho\|_{L^\infty} = \|g\|_{L^\infty}$. Note that $\int_\gamma \rho = +\infty$ for all $\gamma \in \Gamma_1$ and that by Lemma III.2.5 the family of curves $\Gamma_2 = \{\gamma \in \Upsilon : \int_\gamma \rho = +\infty\}$ has ∞-modulus zero. Note that if $\int_\gamma \rho < +\infty$, then the set $(\rho \circ \gamma)^{-1}(+\infty)$ has measure zero in the domain of γ (because otherwise $\int_\gamma \rho = +\infty$). Thus, if $\int_\gamma \rho < +\infty$, we have in particular that $\int_\gamma \rho \leq \|\rho\|_{L^\infty} \ell(\gamma)$. By Theorem III.3.3 there exists a set $F \subset X$ of measure 0 and a constant $K > 0$ (depending only on X) such that for all $x, y \in X \setminus F$ there exist a rectifiable curve γ such that $\int_\gamma \rho < +\infty$ and $\ell(\gamma) \leq Kd(x, y)$. Let now $x, y \in X\backslash F$ and γ be a rectifiable curve satisfying the precedent conditions. Then

$$|f(x) - f(y)| \leq \int_\gamma \rho \leq \|\rho\|_{L^\infty} \ell(\gamma) \leq \|\rho\|_{L^\infty} Kd(x, y).$$

Therefore f is $(\|\rho\|_{L^\infty} K)$-Lipschitz μ-a.e. and using McShane's Extension Theorem one can extend f to a function $\widetilde{f} \in \mathrm{LIP}^\infty(X)$ such that $f = \widetilde{f}$ μ-a.e. In particular, $\widetilde{f} \in \widetilde{N}^{1,\infty}(X)$ and by Lemma III.2.12, $f \sim \widetilde{f}$. Thus, $\mathrm{LIP}^\infty(X) = M^{1,\infty}(X) = N^{1,\infty}(X)$. □

Our purpose now is to see under which conditions the spaces $D^\infty(X)$ and $N^{1,\infty}(X)$ coincide. For that, we need first to use the local version of the weak p-Poincaré inequality (see for example Definition 4.2.17 in [95]).

Definition III.3.6. Let $1 \leq p < \infty$. We say that (X, d, μ) supports a *uniform local weak p-Poincaré inequality* with constant C_p if for every $x \in X$, there exists a neighborhood U_x of x and $\lambda \geq 1$ such that whenever B is a ball in X such that λB is contained in U_x, and f is an integrable function on λB with g as its upper gradient in λB, then

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq C_p r \Big(\fint_{B(x,\lambda r)} g^p d\mu \Big)^{1/p}.$$

Under the hypothesis of the corollary below, it can be checked that a local version of Theorem III.3.3 holds. Keeping this in mind, and pointing out that $\mathrm{Lip}(\cdot)$ depends only on local estimates, the next corollary follows from Corollary III.3.5 together with Lemma III.3.1.

Corollary III.3.7. *Let X be a complete metric space that supports a doubling Borel measure μ which is nontrivial and finite on balls. If X supports a uniform local weak p-Poincaré inequality for $1 \leq p < \infty$. Then $N^{1,\infty}(X) = D^\infty(X)$ with equivalent norms.*

Proof. If $f \in N^{1,\infty}(X)$, then there exists an ∞-weak upper gradient $g \in L^\infty(X)$ of f. We construct in the same way as in Corollary III.3.5 an ∞-weak upper gradient ρ of f which satisfies that $\|\rho\|_\infty = \|g\|_\infty$ and $\int_\gamma \rho = \|\rho\|_\infty \ell(\gamma)$ for all $\gamma \in \Upsilon$ such that $\int_\gamma \rho < \infty$. Fix $x \in X$. By the uniform local weak p-Poincaré inequality together with Theorem III.3.3, there exists a neighborhood U^x such that for almost every $z, y \in U^x$ there exist a rectifiable curve γ such that $\int_\gamma \rho < +\infty$ and $\ell(\gamma) \leq Kd(z, y)$. Here K is the constant which appears in Theorem III.3.3. Let now $z, y \in U^x$ and γ a rectifiable curve satisfying the precedent conditions. Then

$$|f(z) - f(y)| \leq \int_\gamma \rho = \|\rho\|_\infty \ell(\gamma) \leq \|\rho\|_\infty Kd(z, y).$$

Then, we deduce that $f|_{U_x \backslash F}$ is $\|\rho\|_\infty K$-Lipschitz where F is a zero-measure set. Since the Lipschitz constant does not depends on x, f admits a uniform locally $\|\rho\|_\infty K$-Lipschitz extension to the whole U_x, which is, in particular, continuous. Hence, $f \in \mathrm{LIP}^{\mathrm{u},\infty}_{\mathrm{loc}}(X)$ and since

$$\mathrm{LIP}^{\mathrm{u},\infty}_{\mathrm{loc}}(X) \subset D^\infty(X) \subset N^{1,\infty}(X),$$

we have that $f \in D^\infty(X)$. Here $\mathrm{LIP}^{\mathrm{u},\infty}_{\mathrm{loc}}(X)$ denotes the space of uniform locally Lipschitz functions. □

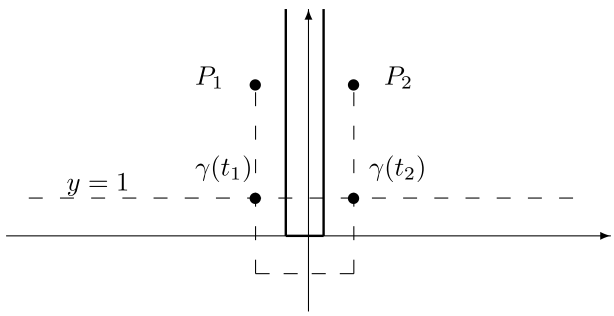
Observe that under the hypothesis of Corollary III.3.7 we have that X is a locally radially quasiconvex metric space. We see throughout a very simple example that in general there exist metric spaces X for which the following holds:

$$\mathrm{LIP}^\infty(X) = M^{1,\infty}(X) \subsetneq D^\infty(X) = N^{1,\infty}(X).$$

The next examples illustrate such fact.

Examples III.3.8.

- Consider the metric space $X = \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R}^2 : -1 < x < 1 \text{ and } 0 < y\}$, endowed with the restriction to X of the Euclidean metric of $\mathbb{R}^2$ and the 2-dimensional Lebesgue measure $\mathscr{L}^2$.



Since X is a complete metric space that supports a doubling measure and a local uniform weak p-Poincaré inequality for any $1 \le p < \infty$, by Corollary III.3.7, we have that $D^\infty(X) = N^{1,\infty}(X)$. Let

$$f(x,y) = \begin{cases} 1 & \text{if } x \ge 0 \text{ or } y \le 0 \\ 0 & \text{if } x \le 0 \text{ and } y \ge 1 \\ 1-y & \text{if } x \le 0 \text{ and } 0 \le y \le 1. \end{cases}$$

One can check that $f \in D^\infty(X) = N^{1,\infty}(X)$. Indeed, let us see that the function

$$g(x,y) = \begin{cases} 0 & \text{if } y \ge 1, \\ 1 & \text{if } y < 1 \end{cases}$$

is an ∞-weak upper gradient of f. Take two points in X, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. If $x_1, x_2 \geq 0$ or $x_1, x_2 \leq 0$ or $x_1 \leq 0$, $x_2 \geq 0$ and $y_1, y_2 \leq 1$ it can be checked straightforwardly that $|f(P_1) - f(P_2)| \leq \int_\gamma g$ for any rectifiable curve γ connecting P_1 and P_2. On the other hand, if $x_1 \geq 0$, $x_2 \leq 0$, and $y_1, y_2 \geq 1$, a rectifiable curve connecting P_1 and P_2 must go down and then go up around the origin. If we denote by

$$t_1 = \max\{t \in I : \gamma(t) = (x,1), x < 0\} \quad \text{and} \quad t_2 = \min\{t \in I : \gamma(t) = (x,1), x > 0\},$$

we have that

$$\int_\gamma g \geq \int_{t_1}^{t_2} g(\gamma(t)) \geq \int_{t_1}^{t^2} 1 \overset{(*)}{>} 2 + 1/2 > 1 = |f(P_1) - f(P_2)|$$

for each γ parametrized by arc-length connecting P_1 and P_2. Looking at the figure above, since the curve must border the origin the length $\int_{t_1}^{t^2} 1$ is strictly greater than $2+1/2$, and so inequality $(*)$ holds. Thus, g is an upper gradient of f and so an ∞-weak upper gradient for f. However, $f \notin \mathrm{LIP}^\infty(X)$, and so $\mathrm{LIP}^\infty(X) \subsetneq D^\infty(X) = N^{1,\infty}(X)$. Notice that this space does not admit a global p-Poincaré inequality for any p. For example, if we consider the point $P_t = (-1-1/2, t)$ and the ball $B(P_t, 3)$,

$$\int_{\lambda B(P_t,3)} g \, d\mathscr{L}^2 = 0,$$

for t large enough because λ is constant. However, $\int_{B(P_t,3)} |f - f_{B(P_t,3)}| \, d\mathscr{L}^2 > 0$, and so, the p-Poincaré inequality does not hold for any p.

- Consider the metric space (X, d) where $X = \mathbb{C} \backslash \{(x, 0) : x \geq 0\}$ and d the metric induced by the Euclidean one, which is a locally compact metric space. Let $f(z) = \arg(z)$, for each $z \in X$. One can check that $f \in N^{1,\infty}(X) \backslash \mathrm{LIP}^\infty(X)$. Observe that since X is locally isomorphic to $\mathbb{R}^2$ admits a local uniform weak p-Poincaré inequality; hence, by Corollary III.3.7 (see also Remark III.3.4), $D^\infty(X) = N^{1,\infty}(X)$.

Chapter IV

∞-Poincaré inequality in metric measure spaces

A nice feature of the Euclidean n-space, $n \geq 2$ is the fact that every pair of points x and y can be joined not only by the line segment $[x, y]$, but also by a large family of curves whose length is comparable to the distance between the points. Once one has found such a "thick" family of curves, the deduction of important Sobolev and Poincaré inequalities is an abstract procedure that only uses some techniques in analysis in which the Euclidean structure no longer plays a role.

The classical Poincaré inequality allows one to obtain integral bounds on the oscillation of a function using integral bounds on its derivatives. It is worth mentioning that in this type of inequalities the derivative itself is not needed, but only the size of the gradient of the function is really used; a nice discussion of this can be found in [93]. This is the idea behind many generalizations of Poincaré inequalities, in spaces where we may not have a linear structure. The idea of Poincaré inequalities makes sense in the more general setting of metric measure spaces as we have already seen in Definition III.3.2.

There is a long list of metric spaces supporting a Poincaré inequality, including some standard examples such as $\mathbb{R}^n$, Riemannian manifolds with nonnegative Ricci curvature, Carnot groups (in particular the Heisenberg group), but also other nonRiemannian metric measure spaces of fractional Hausdorff dimension, see for example [80], [55] and references therein. Metric spaces equipped with a Poincaré inequality support a nontrivial potential theoretic and geometric theory even without a priori smoothness structure of the metric space. Moreover, they admit a first order differential calculus theory akin to that in Euclidean spaces. One surprising fact is that some geometric consequences of this condition seem to be independent of the parameter p and the picture is not yet clear.

It follows from Hölder's inequality that if a space admits a p-Poincaré inequality, then it admits a q-Poincaré inequality for each $q \geq p$. Recently Keith and Zhong [73] proved a self-improving property for Poincaré inequalities, that is, if X is a complete metric space equipped with a doubling measure satisfying a p-Poincaré inequality for some $1 < p < \infty$, then there exists $\varepsilon > 0$ such that X supports a q-Poincaré inequality for all $q > p - \varepsilon$. The strongest of all these inequalities would be the 1-Poincaré inequality, which is closely related to relative isoperimetric inequalities. For example, it is well known that the 1-Poincaré inequality is equivalent to the relative isoperimetric property [87], [13]. On the other hand, even for $p > 1$ the p-Poincaré inequality has strong links with the geometry of the underlying metric measure space. For instance, the Poincaré inequality implies that any pair of points in the space can be connected by many curves that are not too long; this property is called quasiconvexity. A natural question is what would be the weakest version of p-Poincaré inequality that would still give reasonable information on the geometry of the metric space. One of the goals of this chapter is to answer this question, by studying a version of ∞-Poincaré inequality (see Definition IV.1.1).

The main result of this chapter is a characterization of spaces supporting an ∞-Poincaré inequality; this is given in Theorem IV.2.8. A metric measure space is said to be *thick quasiconvex* if, loosely speaking, every pair of sets of positive measure, which are a positive distance apart, can be connected by a "thick" family of quasiconvex curves in the sense that the ∞-modulus of this family of curves is positive. The first aim is to show that a connected complete doubling metric measure space supports a weak ∞-Poincaré inequality if and only if it is thick quasiconvex, which is a purely geometric condition. We will also prove that this condition is equivalent to the purely analytic condition that $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ with equivalent energy norms, that is, every Lipschitz function belongs to an equivalence class in $N^{1,\infty}(X)$ and every function in any equivalence class in $N^{1,\infty}(X)$ can be modified on a set of measure zero to become a Lipschitz continuous function.

In Section IV.1 we introduce the ∞-Poincaré inequality and present an example (Example IV.1.4) of a nondoubling metric space which supports an ∞-Poincaré inequality but does not support any p-Poincaré inequality for $p < \infty$. Furthermore, we give some geometric implications of the ∞-Poincaré inequality, namely, that the space is quasiconvex. However, as one can appreciate in Corollary IV.2.16, quasiconvexity is not a sufficient condition for a space to support an ∞-Poincaré inequality. In Section IV.2 we will introduce the stronger notion of thick quasiconvexity (Definition IV.2.1), which leads us in Theorem IV.2.8 to obtain the desired analytic and geometric characterization of ∞-Poincaré inequality.

In Section IV.3 we will point out some of the differences between the consequences of p-Poincaré inequality and that of ∞-Poincaré inequality. These differences appear to be due to the fact that unlike the L^p-norm for finite p, the L^∞-norm is not sensitive to small local perturbations. We study a concept analogous to thick quasiconvexity associated with p-Poincaré inequality for finite $p \geq 1$, *p-thick quasiconvextiy* (see Definition IV.2.1) and prove in Proposition IV.3.5 that spaces supporting a p-Poincaré inequality are p-thick quasiconvex. We also give an example (Example IV.3.6) which illustrates that this analogous geometric property (p-thick quasiconvexity) does not imply the validity of a p-Poincaré inequality. The metric measure space given in this example is doubling and supports an ∞-Poincaré inequality, but supports no finite p-Poincaré inequality. So this example shows in addition that one cannot expect a self-improving property for ∞-Poincaré inequalities in the spirit of Keith and Zhong [73].

We will finish Chapter IV discussing the persistence of ∞-Poincaré inequalities under Gromov-Hausdorff convergence. The discussion in Chapter 9 of [24] demonstrates that if

$\{X_n, d_n, \mu_n\}_n$ is a sequence of metric measure spaces with μ_n doubling measures supporting a p-Poincaré inequality, and in addition the constants associated with the doubling property and Poincaré inequality are uniformly bounded, and furthermore, this sequence of metric measure spaces converges in the measured Gromov-Hausdorff sense to a metric measure space (X, d, μ), then this limit space also is doubling and supports a p-Poincaré inequality. We will provide an example (Example IV.3.9) which demonstrates that this persistence of Poincaré inequality under measured Gromov-Hausdorff limits fails for ∞-Poincaré inequality.

IV.1 ∞-Poincaré inequality in metric measure spaces

We study the following version of ∞-Poincaré inequality.

Definition IV.1.1. We say that (X, d, μ) supports a *weak ∞-Poincaré inequality* if there exist constants $C > 0$ and $\lambda \geq 1$ such that for every Borel measurable function $f : X \to \mathbb{R} \cup \{\infty, -\infty\}$ and every upper gradient $g : X \to [0, \infty]$ of f, the pair (f, g) satisfies the inequality

$$\fint_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \leq C\, r \|g\|_{L^\infty(B(x,\lambda r))}$$

for each ball $B(x, r) \subset X$.

Remark IV.1.2. Let us observe that

$$\begin{aligned}\fint_B |f(x) - f_B| \, d\mu(x) &= \fint_B \Big| \fint_B (f(x) - f(y)) d\mu(y) \Big| d\mu(x) \\ &\leq \fint_B \fint_B |f(x) - f(y)| d\mu(y) d\mu(x),\end{aligned}$$

and so, when we want to check that (X, d, μ) supports a weak ∞-Poincaré inequality, it is enough to prove that each pair (f, g) satisfies

$$\fint_B \fint_B |f(x) - f(y)| d\mu(y) d\mu(x) \leq C\, r \|g\|_{L^\infty(\lambda B)} \tag{IV.1}$$

for each ball $B \subset X$ with radius r. On the other hand, the inequality (IV.1) is necessary to verify ∞-Poincaré inequality as well. To see this, note that

$$\begin{aligned}\fint_B \fint_B |f(x) - f(y)| d\mu(y) d\mu(x) &= \fint_B \fint_B |f(x) - f_B + f_B - f(y)| d\mu(y) d\mu(x) \\ &\leq 2 \fint_B |f(x) - f_B| d\mu(x).\end{aligned}$$

Another useful inequality is the following. Given a measurable function f in X,

$$\fint_B |f - f_B| \, d\mu \leq 2 \inf_{c \in \mathbb{R}} \fint_B |f - c| d\mu. \tag{IV.2}$$

Indeed, let $c \in \mathbb{R}$ and suppose $c \geq f_B$ (the case $c < f_B$ is analogous). Then,

$$\fint_B |c - f_B|\, d\mu = c - f_B = \fint_B c - \fint_B f = \fint_B (c - f) \leq \fint_B |c - f| d\mu.$$

Since $|f(x) - f_B| \leq |f(x) - c| + |c - f_B|$ for each $x \in X$, we have that

$$\fint_B |f - f_B|\, d\mu \leq \fint_B |f - c|\, d\mu + \fint_B |c - f_B|\, d\mu \leq 2 \fint_B |f - c| d\mu.$$

If we take the infimum over c on the right hand of the previous inequality, we get inequality (IV.2).

Remark IV.1.3. If the measure is doubling, the ∞-Poincaré inequality is qualitatively invariant under bi-Lipschitz mappings. Indeed, let (X, d, μ) be a metric measure space with μ doubling which supports a weak ∞-Poincaré inequality and $(\widetilde{X}, \widetilde{d})$ be another metric space which is bi-Lipschitz equivalent to (X, d), so there exists $T : X \to \widetilde{X}$ which is L-bi-Lipschitz. We denote by $\widetilde{\mu}$ the push-forward measure of μ by T, that is, $\widetilde{\mu}(A) = \mu(T^{-1}(A))$ for each subset $A \subset \widetilde{X}$. Observe that μ and $\widetilde{\mu}$ have the same zero measure sets.

Fix $\widetilde{x} \in \widetilde{X}$, $r > 0$ and a function $\widetilde{f} : \widetilde{X} \to \mathbb{R}$. We adopt the notation $x = T^{-1}(\widetilde{x})$, $f = \widetilde{f} \circ T$ and $\widetilde{B}(\widetilde{x}, r) = \{y \in \widetilde{X} : \widetilde{d}(y, \widetilde{x}) < r\}$. We have that $T^{-1}(\widetilde{B}(\widetilde{x}, r)) = \{y \in X : \widetilde{d}(\widetilde{x}, T(y)) < r\}$ and since T is L-bi-Lipschitz,

$$B(x, \frac{r}{L}) \subset T^{-1}(\widetilde{B}(\widetilde{x}, r)) \subset B(x, Lr). \tag{IV.3}$$

Therefore,

$$\begin{aligned}
\fint_{\widetilde{B}(\widetilde{x},r)} |\widetilde{f} - \widetilde{f}_{\widetilde{B}(\widetilde{x},r)}|\, d\widetilde{\mu} &\overset{\text{(IV.2)}}{\leq} 2 \inf_{c \in \mathbb{R}} \fint_{\widetilde{B}(\widetilde{x},r)} |\widetilde{f} - c|\, d\widetilde{\mu} \\
&\leq \frac{2}{\widetilde{\mu}(\widetilde{B}(\widetilde{x}, r))} \int_{\widetilde{B}(\widetilde{x},r)} |\widetilde{f} - f_{B(x,Lr)}|\, d\widetilde{\mu} \\
&= \frac{2}{\mu(T^{-1}(\widetilde{B}(\widetilde{x}, r)))} \int_{T^{-1}(\widetilde{B}(\widetilde{x},r))} |f - f_{B(x,Lr)}|\, d\mu \\
&\leq \frac{2}{\mu(T^{-1}(\widetilde{B}(\widetilde{x}, r)))} \frac{\mu(B(x, Lr))}{\mu(B(x, Lr))} \int_{B(x,Lr)} |f - f_{B(x,Lr)}|\, d\mu \quad \text{(IV.3)} \\
&\leq 2\, \frac{\mu(B(x, Lr))}{\mu(B(x, \frac{r}{L}))} \fint_{B(x,Lr)} |f - f_{B(x,Lr)}| d\mu \quad \text{(IV.3)} \\
&\leq 2\, C_{\mu,L}^2 \fint_{B(x,Lr)} |f - f_{B(x,Lr)}|\, d\mu,
\end{aligned}$$

where $C_{\mu,L}$ is a constant which satisfies $\mu(LB) \leq C_{\mu,L}\, \mu(B)$. Since (X, d, μ) supports an

∞-Poincaré inequality,

$$\begin{aligned}\fint_{\widetilde{B}(\widetilde{x},r)} |\widetilde{f} - \widetilde{f}_{B(\widetilde{x},r)}| d\widetilde{\mu} &\le 2C^2_{\mu,L} \fint_{B(x,Lr)} |f - f_{B(x,Lr)}| d\mu \\ &\le 2C^2_{\mu,L} CLr \|g_f\|_{L^\infty(B(x,\lambda Lr))} \\ &\le 2C^2_{\mu,L} CLr \|g_{\widetilde{f}}\|_{L^\infty(T(B(x,\lambda Lr)))} \\ &\le 2C^2_{\mu,L} CLr \|g_{\widetilde{f}}\|_{L^\infty(\widetilde{B}(\widetilde{x},\lambda L^2 r))},\end{aligned}$$

where g_f and $g_{\widetilde{f}}$ denote upper gradients for f and $\widetilde{f}$ respectively.

Thus, $(\widetilde{X}, \widetilde{d}, \widetilde{\mu})$ satisfies an ∞-Poincaré inequality with constants $\widetilde{C} = 2C^2_{\mu,L} CL$ and $\widetilde{\lambda} = \lambda L^2$.

The next example shows that there exist spaces with a weak ∞-Poincaré inequality which do not admit a weak p-Poincaré inequality for any finite p.

Example IV.1.4. Let T be a nondegenerate triangular region in $\mathbb{R}^2$ and let T' be an identical copy of T. Let X be the metric space obtained by identifying a vertex V of T with a vertex V' of T' ($V = V' = \{0\}$) and the metric defined by

$$d(x,y) = \begin{cases} |x-y| & \text{if } x,y \in T \text{ or } x,y \in T' \\ |x-V| + |V'-y| & \text{if } x \in T \text{ and } y \in T'. \end{cases}$$

The space is equipped with the weighted measure μ given by $d\mu(x) = \omega(x) d\mathscr{L}^2(x)$, where $\omega(x) = e^{-\frac{1}{|x|^2}}$. Note that μ and the Lebesgue measure $\mathscr{L}^2$ have the same zero measure sets. It is already known that this space equipped with the Lebesgue measure $\mathscr{L}^2$ admits a p-Poincaré inequality for $p > 2$ (see Example III.1.4 or [95]). Let us see that (X, d, μ) does not admit a weak p-Poincaré inequality for any finite p but admits a weak ∞-Poincaré inequality.

Let us consider an upper gradient g of f. Now, we obtain the following chain of inequalities by using Hölder's inequality for $2 < p < q$:

$$\begin{aligned}\fint_B |f - f_B|\, d\mu &\overset{(\mathrm{IV.2})}{\le} 2 \inf_{c \in \mathbb{R}} \fint_B |f - c| d\mu \le 2 \fint_B |f - f_{B,\mathscr{L}^2}| d\mu \\ &\le 2\|f - f_{B,\mathscr{L}^2}\|_{L^\infty(\mu)} = 2\|f - f_{B,\mathscr{L}^2}\|_{L^\infty(\mathscr{L}^2)} \\ &\le C_p r \Big(\fint_{5\lambda B} g^p d\mathscr{L}^2 \Big)^{1/p} \le C_p r \Big(\fint_{5\lambda B} g^q d\mathscr{L}^2 \Big)^{1/q},\end{aligned}$$

where $f_{B,\mathscr{L}^2} = \fint_B f d\mathscr{L}^2$. In the third line of the previous chain of inequalities we have applied [54, Theorem 5.1]. If we let q tend to infinity we get

$$\fint_B |f - f_B|\, d\mu \le C_p r \|g\|_{L^\infty(\mathscr{L}^2, 5\lambda B)} = C_p r \|g\|_{L^\infty(\mu, 5\lambda B)},$$

and so, (X, d, μ) admits a weak ∞-Poincaré inequality.

Let us see now that (X, d, μ) does not admit a p-Poincaré inequality for any finite p. Indeed, consider the function $f = 1$ in $T \setminus \{V\}$ and $f = 0$ in T' and in the vertex V. It is not difficult to check that the function $g_\alpha(x) = \frac{\alpha}{|x|}$ is an upper gradient for f for each $\alpha > 0$. Taking the ball $B = X$, we have that $f_X > 0$ and therefore $⨍_X |f - f_X|\, d\mu > 0$. Nevertheless, $\int_X g_\alpha^p d\mu$ tends to zero when α tends to zero for $1 < p < \infty$, and so X does not admit a weak p-Poincaré inequality for any finite p.

Observe that the measure μ in the above example is *not* doubling.

One of the most useful geometric implications of the p-Poincaré inequality for finite p is the fact that if a complete doubling metric measure space supports a p-Poincaré inequality then there exists a constant such that each pair of points can be connected with a curve whose length is at most the constant times the distance between the points (see [94] or [54]), that is, the space is *quasiconvex*. If X is only known to support an ∞-Poincaré inequality, the same conclusion holds as demonstrated by Proposition IV.1.5 below.

Proposition IV.1.5. *Suppose that (X, d, μ) is a complete metric measure space with μ a doubling measure. If X supports a weak ∞-Poincaré inequality, then X is quasiconvex with a constant depending only on the constants of the Poincaré inequality and the doubling constant.*

Proof. Let $\varepsilon > 0$. We say that $x, z \in X$ lie in the same ε-component of X if there exists an ε-chain joining x with z, that is, there exists a finite chain $z_0, z_1, \ldots, z_n$ such that $z_0 = x$, $z_n = z$ and $d(z_i, z_{i+1}) \leq \varepsilon$ for all $i = 0, \ldots, n-1$. If x and y lie in different ε-components, then it is obvious that there does not exist a rectifiable curve joining x and y. Thus, the function $g \equiv 0$ is an upper gradient for the characteristic function of any of the components. Note that for every x in one of the components, the ball $B(x, \varepsilon/2)$ is a subset of that component; that is, each component is open and hence is a measurable set. By applying the weak ∞-Poincaré inequality to the characteristic function of any component, it follows that all the points of X lie in the same ε-component.

Now, let us fix $x, y \in X$ and prove that there exists a curve γ joining x and y such that $\ell(\gamma) \leq C d(x, y)$, where C is a constant which depends only on the doubling constant and the constants involved in the Poincaré inequality. We recall here the definition of ε*-distance* of x to z to be

$$\rho_{x,\varepsilon}(z) := \inf \sum_{i=0}^{N-1} d(z_i, z_{i+1}),$$

where the infimum is taken over all finite ε-chains $(z_i)_{i=0}^N$ joining $z_0 = x$ to $z_N = z$. Note that $\rho_{x,\varepsilon}(z) < \infty$ for all $z \in X$. In addition, if $d(z, w) \leq \varepsilon$ then $|\rho_{x,\varepsilon}(z) - \rho_{x,\varepsilon}(w)| \leq d(z, w)$. Hence, $\rho_{x,\varepsilon}$ is a locally 1-Lipschitz function, in particular, every point is a Lebesgue point of $\rho_{x,\varepsilon}$. Moreover, for all $\varepsilon > 0$, the function $g \equiv 1$ is an upper gradient of $\rho_{x,\varepsilon}$. For each

$i \in \mathbb{Z}$, define $B_i = B(x, 2^{1-i}d(x,y))$ if $i \geq 0$, and $B_i = B(y, 2^{1+i}d(x,y))$ if $i \leq -1$. Thus, a telescopic argument, together with weak ∞-Poincaré inequality, gives us the following chain of inequalities:

$$\begin{aligned}
|\rho_{x,\varepsilon}(y)| =& |\rho_{x,\varepsilon}(x) - \rho_{x,\varepsilon}(y)| \\
\leq & \sum_{i\in\mathbb{Z}} \Big| \fint_{B_i} \rho_{x,\varepsilon} d\mu - \fint_{B_{i+1}} \rho_{x,\varepsilon} d\mu \Big| \\
\leq & C_\mu \sum_{i\in\mathbb{Z}} \frac{1}{\mu(B_i)} \fint_{B_i} \Big|\rho_{x,\varepsilon} - \fint_{B_{i+1}} \rho_{x,\varepsilon} d\mu\Big| d\mu \\
\leq & C_\mu C d(x,y) \sum_{i\in\mathbb{Z}} 2^{-|i|} \|g\|_{L^\infty(\lambda B_i)} \\
\leq & C d(x,y)
\end{aligned} \tag{IV.4}$$

where C is a constant that depends only on X.

Since X is complete, the existence of a nontrivial doubling measure implies that closed balls are compact. Using a standard limiting argument, which involves Arzela-Ascoli's Theorem and Inequality (IV.4), we can construct a 1-Lipschitz rectifiable curve connecting x and y with length at most $Cd(x,y)$. Since x and y were arbitrary this completes the proof. For further details about the construction of the curve we refer the reader to [79, Theorem 3.1]. $\square$

The following technical lemma will be useful in the sequel. The proof of the fact that f is μ-measurable follows the lines of [65, Corollary 1.10]. We include it here for the sake of completeness.

Lemma IV.1.6. *Let X be a complete separable metric space equipped with a σ-finite Borel measure μ, and let $g : X \longrightarrow [0,\infty]$ be a Borel function. Then, for each $x_0 \in X$ and $r > 0$, the function*

$$f(z) = \inf_{\gamma \text{ connects } z \text{ to } B(x_0,r)} \int_\gamma g\, ds,$$

is μ-measurable. Moreover, whenever $k \in \mathbb{R}$ the function g is an upper gradient for $\tilde{f} = \min\{f,k\}$.

Proof. Let $\mathcal{P}$ be the set of all paths $\gamma : [0,1] \to X$ equipped with the metric

$$d_\infty(\gamma, \tilde{\gamma}) = \sup_{t\in[0,1]} d(\gamma(t), \tilde{\gamma}(t)),$$

which is a complete separable metric space (observe that we consider all paths, not only rectifiable, since the space of rectifiable paths under the supremum norm is not complete). Let $\mathcal{P}_{B(x_0,r)} \subset \mathcal{P}$ be the set of all paths starting from $B(x_0,r)$ which is an open set. By

Lemma [65, 2.2] applied to the function $\rho = 1$, the function $\phi : \mathcal{P} \to [0, \infty]$, $\phi(\gamma) = \ell(\gamma)$ is lower semicontinuous. Therefore,

$$G = \bigcup_{x \in X} \Gamma_{B(x_0,r),x} = \phi^{-1}([0, \infty)) \cap \mathcal{P}_{B(x_0,r)},$$

is a Borel set. Here $\Gamma_{B(x_0,r)}, x$ is the set of all rectifiable paths connecting $B(x_0, r)$ to x.

Consider the mapping $\varphi_g : \mathcal{P}_{B(x_0,r)} \to [0, \infty]$ defined by $\varphi_g(\gamma) = \int_\gamma g ds$. The proof of Theorem [65, 1.8] gives that φ_g is a Borel function. Define also $\pi : \mathcal{P}_{B(x_0,r)} \to X$, as $\pi(\gamma) = \gamma(1)$ for all $\gamma \in \mathcal{P}_{B(x_0,r)}$. The choice of the metric in $\mathcal{P}_{B(x_0,r)}$ guarantees that π is continuous. We have that, for all real numbers $a > 0$,

$$f^{-1}([0, a)) = \pi(\varphi_g^{-1}([0, a)) \cap G),$$

so by the above considerations it is the continuous image of a Borel set. By Lusin's theorem (see [68, 21.10]), analytic subsets (in particular a continuous image of a Borel set) of a complete separable metric space X are μ-measurable for any σ-finite Borel measure μ on X and the result follows.

To see that g is an upper gradient of $\tilde{f}$ on X, we argue as follows. Fix $z_1, z_2 \in X$ and β be a rectifiable curve in X connecting z_1 to z_2. There are three possible cases:

1. $\tilde{f}(z_1) = f(z_1)$ and $\tilde{f}(z_2) = f(z_2)$,
2. $\tilde{f}(z_1) = f(z_1)$ and $\tilde{f}(z_2) = k$,
3. $\tilde{f}(z_1) = k = \tilde{f}(z_2)$.

In the first case, both $f(z_1)$ and $f(z_2)$ are finite. Fix $\varepsilon > 0$; then we can find a rectifiable curve connecting z_1 to $B(x, \varepsilon)$ such that $f(z_1) \geq \int_\gamma g ds - \varepsilon$, and so

$$f(z_2) - f(z_1) \leq \int_{\gamma \cup \beta} g\, ds - \int_\gamma g\, ds + \varepsilon = \int_\beta g\, ds + \varepsilon,$$

where we can cancel $\int_\gamma g\, ds$ because it is a finite value. A similar argument gives

$$f(z_1) - f(z_2) \leq \int_\beta g\, ds + \varepsilon,$$

and the combination of the above two inequalities followed by letting $\varepsilon \to 0$ gives

$$|\tilde{f}(z_1) - \tilde{f}(z_2)| = |f(z_1) - f(z_2)| \leq \int_\beta g\, ds.$$

In the second case, $f(z_1) = f(z_1) \le \tilde{f}(z_2) \le f(z_2)$. In this case again, $f(z_1)$ is finite. For $\varepsilon > 0$ we can find a rectifiable curve γ connecting z_1 to $B(x, \varepsilon)$ such that $f(z_1) \ge \int_\gamma g\,ds - \varepsilon$, and so

$$
\begin{aligned}
|\tilde{f}(z_1) - \tilde{f}(z_2)| = \tilde{f}(z_2) - \tilde{f}(z_1) \le f(z_2) - f(z_1) &\le \int_{\gamma\cup\beta} g\,ds - \int_\gamma g\,ds + \varepsilon \\
&= \int_\beta g\,ds + \varepsilon,
\end{aligned}
$$

where again we were able to cancel the term $\int_\gamma g\,ds \le u(z_1) + \varepsilon$ because it is finite. Letting $\varepsilon \to 0$ we again obtain

$$
|\tilde{f}(z_1) - \tilde{f}(z_2)| \le \int_\beta g\,ds.
$$

In the third case we easily obtain the above inequality again, because in this case $\tilde{f}(z_1) - \tilde{f}(z_2) = 0$. $\square$

The following example shows one of the difficulties in working with $p = \infty$ as opposed to finite values of p.

Example IV.1.7. Let X be a complete metric space that supports a doubling Borel measure μ which is nontrivial and finite on balls, and suppose that X supports a weak ∞-Poincaré inequality. Denote by $\Gamma_{x_0,r,R}$ the family of curves that connect $B(x_0, r)$ to the complement of the ball $B(x_0, R)$ with $0 < r < R/2 < \operatorname{diam}(X)/4$.

We will prove that there is a constant $C > 0$, independent of R, r and x_0, such that

$$
\operatorname{Mod}_\infty(\Gamma_{x_0,r,R}) \ge \frac{C}{R}.
$$

To see this, let g be a nonnegative Borel measurable function on X such that for all $\gamma \in \Gamma_{x_0,r,R}$, the integral $\int_\gamma g\,ds \ge 1$. Notice here that by Proposition IV.1.5, X is quasiconvex. We then set

$$
\tilde{f}(z) = \inf_{\gamma \text{ path connecting } z \text{ to } B(x_0,r)} \int_\gamma g\,ds,
$$

and consider $f = \min\{\tilde{f}, 2\}$. Then it follows that $f = 0$ on $B(x_0, r)$ and by the choice of g, $f \ge 1$ on $X \setminus B(x_0, R)$. By Lemma IV.1.6 it follows that f is measurable and that g is an upper gradient of f; that is, $f \in N^{1,\infty}(X)$.

If $x \in B(x_0, r)$ and $y \in B(x_0, R + r) \setminus B(x_0, R)$, $\forall i \in \mathbb{Z}$ define $B_i = B(x, 2^{1-i} d(x, y))$ if $i \ge 0$, and $B_i = B(y, 2^{1+i} d(x, y))$ if $i \le -1$. By the weak ∞-Poincaré inequality and the doubling property of μ, we get for Lebesgue points $x \in B(x_0, r)$ and $y \in X \setminus B(x_0, R)$,

$$\begin{aligned}
1 \leq |f(x) - f(y)| &\leq \sum_{i \in \mathbb{Z}} \Big| \fint_{B_i} f \, d\mu - \fint_{B_{i+1}} f \, d\mu \Big| \\
&\leq C_\mu \sum_{i \in \mathbb{Z}} \fint_{B_i} \Big| f - \fint_{B_i} f \, d\mu \Big| d\mu \\
&\leq C_\mu C d(x, y) \sum_{i \in \mathbb{Z}} 2^{-|i|} \|g\|_{L^\infty(\lambda B_i)} \\
&\leq C d(x, y) \|g\|_{L^\infty(X)}.
\end{aligned}$$

Hence

$$\|g\|_{L^\infty(X)} \geq \frac{1}{C\, d(x, y)} \geq \frac{1}{C\,(R + r)} \geq \frac{1}{2CR}.$$

Taking the infimum over all such g we obtain the desired inequality for the ∞-Modulus. An analogous statement holds for $\mathrm{Mod}_p(\Gamma_{x_0,r,R})$ if X supports a weak p-Poincaré inequality for sufficiently large finite p (that is, with p larger than the lower mass bound exponent s_1 obtained from the doubling property of the measure μ). For such finite p, we can approximate test functions g from above in $L^p(X)$ by lower semi-continuous functions (it follows from Vitali-Caratheodory theorem [42, pp. 209–213]), and so we would see as in [59] that the p-modulus of the collection of all curves that connect x_0 itself to $X \setminus B(x_0, R)$ is positive. Unfortunately such an approximation by lower semi-continuous functions in the L^∞-norm does not hold true, and so we cannot conclude from the above computation that the ∞-modulus of the collection of all curves connecting x_0 to $X \setminus B(x_0, R)$ is positive *if X is only known to support a weak ∞-Poincaré inequality.*

The previous example highlights the difficulties when working with the L^∞-norm, namely, the L^∞-norm is insensitive to local changes, and we do not have Vitali-Caratheodory theorem.

IV.2 Geometric characterization of ∞-Poincaré inequality

The connection between isoperimetric and Sobolev-type inequalities in the Euclidean setting is well-understood (see [87], [13]). In the context of metric spaces supporting a doubling measure, Miranda proved in [87] that a 1-weak Poincaré inequality implies a relative isoperimetric inequality for sets of finite perimeter. Recently, in [75] Kinnunen and Korte gave further characterizations of Poincaré type inequalities in the context of Newtonian spaces in terms of isoperimetric and isocapacitary inequalities.

In what follows, we will prove that ∞-Poincaré inequality also has a geometric characterization, namely, it is equivalent to *thick quasiconvexity*.

Definition IV.2.1. (X, d, μ) is a *thick quasiconvex* space if there exists $C \geq 1$ such that for all $x, y \in X$, $0 < \varepsilon < \frac{1}{4}d(x, y)$, and all measurable sets $E \subset B(x, \varepsilon)$, $F \subset B(y, \varepsilon)$ satisfying $\mu(E)\mu(F) > 0$ we have that

$$\mathrm{Mod}_\infty(\Gamma(E, F, C)) > 0,$$

where $\Gamma(E, F, C)$ denotes the set of curves $\gamma_{p,q}$ connecting $p \in E$ and $q \in F$ with $\ell(\gamma_{p,q}) \leq Cd(p, q)$. Here we do not require quantitative control on the modulus of the curve family.

Remark IV.2.2. Note that every complete thick quasiconvex space X supporting a doubling measure is quasiconvex. Indeed, let $x, y \in X$ and choose a sequence ε_j which tends to zero. Since X is thick quasiconvex, there exists a constant $C \geq 1$ such that for every ε_j there exists $x_j \in B(x, \varepsilon_j)$ and $y_j \in B(y, \varepsilon_j)$ and a curve γ_j connecting x_j to y_j with $\ell(\gamma_j) \leq Cd(x_j, y_j)$. Thus, we obtain a sequence $\{\gamma_j\}$ of curves such that

$$\ell(\gamma_j) \leq Cd(x_j, y_j) \leq 2Cd(x, y),$$

that is, a sequence of curves with uniformly bounded length. Since X is a complete doubling metric space and therefore proper, we may use Arzela-Ascoli's theorem to obtain a subsequence, also denoted $\{\gamma_j\}$, which converges uniformly to a curve γ which connects x and y with

$$\ell(\gamma) = \lim_{j\to\infty} \ell(\gamma_j) \leq C \lim_{j\to\infty} d(x_j, y_j) = 2Cd(x, y).$$

However, the converse is not true. In Example IV.2.15 we will give a quasiconvex space endowed with a doubling measure which is not thick quasiconvex.

Standard assumptions : In what follows, we will assume that X is a connected complete metric space supporting a doubling Borel measure μ which is nontrivial and finite on balls.

Remark IV.2.3. The hypothesis of completeness is not so restrictive. The completion $(\hat{X}, \hat{d})$ of a metric space (X, d) is unique up to isometry. Note that (X, d) is a subspace of $(\hat{X}, \hat{d})$ and X is dense in $\hat{X}$. For our purposes, the crucial observation is that the essential features of X are inherited by $\hat{X}$. Indeed, if X is locally complete and there is a doubling Borel measure μ which is nontrivial and finite on balls, we may extend this measure to $\hat{X}$ so that $\hat{X} \setminus X$ has zero measure and the extended measure has the same properties as the original one. Also, if X supports a weak p-Poincaré inequality for some $1 \leq p \leq \infty$, then so does $\hat{X}$. See also [65] for further discussions on this topic.

We have already proved in Proposition IV.1.5 that weak ∞-Poincaré inequality for Lipschitz functions implies quasiconvexity. However, in the following proposition we prove that weak ∞-Poincaré inequality for Newtonian functions implies the stronger property of thick quasiconvexity.

Proposition IV.2.4. *If X supports a weak ∞-Poincaré inequality for functions in $N^{1,\infty}(X)$ with upper gradients in $L^\infty(X)$, then X is thick quasiconvex.*

We wish to point out here that $N^{1,\infty}(X)$ consists precisely of functions in $L^\infty(X)$ that have an upper gradient in $L^\infty(X)$.

Proof. Let $x, y \in X$ such that $x \neq y$, and let $0 < \varepsilon < d(x,y)/4$. First, let us consider the case that $E = B(x,\varepsilon)$ and $F = B(y,\varepsilon)$. Fix $n \in \mathbb{N}$ and let $\Gamma_n = \Gamma(B(x,\varepsilon), B(y,\varepsilon), n)$ be the collection of all rectifiable curves connecting $B(x,\varepsilon)$ to $B(y,\varepsilon)$ such that $\ell(\gamma) \leq n\, d(x,y)$. Observe that by the choice of ε, if p, q are the end points of γ, then $d(p,q)/4 \leq d(x,y) \leq 4d(p,q)$.

Suppose that $\mathrm{Mod}_\infty(\Gamma_n) = 0$. By Lemma III.2.5 there exists a nonnegative Borel measurable function $g \in L^\infty(X)$ such that $\|g\|_{L^\infty(X)} = 0$ and for all $\gamma \in \Gamma_n$, the path integral $\int_\gamma g\, ds = \infty$. In this case we define

$$f(z) = \inf_{\gamma \text{ connecting } z \text{ to } B(x,\varepsilon)} \int_\gamma (1+g)\, ds.$$

Observe that $\|1+g\|_{L^\infty(X)} = 1$ and $f = 0$ on $B(x,\varepsilon)$. If $z \in B(y,\varepsilon)$ and γ is a rectifiable curve connecting z to $B(x,\varepsilon)$, then either $\gamma \in \Gamma_n$ in which case $\int_\gamma (1+g)\, ds \geq \int_\gamma g\, ds = \infty$, or else $\gamma \notin \Gamma_n$, in which case $\ell(\gamma) > nd(x,y)$ and so $\int_\gamma (1+g)\, ds \geq \int_\gamma 1\, ds > nd(x,y)$. Therefore, $f(z) \geq n\, d(x,y)$. It follows that the function $\tilde{f} = \min\{f, 2n\, d(x,y)\}$ has the properties that

1. $\tilde{f} = 0$ on $B(x,\varepsilon)$,
2. $\tilde{f} \geq nd(x,y)$ on $B(y,\varepsilon)$,
3. $\tilde{f} \in N^{1,\infty}(X)$,
4. $1+g$ is an upper gradient of $\tilde{f}$ on X (see Lemma IV.1.6), with $\|g\|_{L^\infty(X)} = 0$.

Let $y_0 \in B(y,\varepsilon/2)$ be a Lebesgue point of $\tilde{f}$; then by considering the chain of balls $B_i = B(x, 2^{1-i}d(x,y))$ if $i \geq 0$ and $B_i = B(y_0, 2^{1+i}d(x,y))$ if $i \leq -1$ and using the weak ∞-Poincaré inequality, we get

$$\begin{aligned}
n\, d(x,y) \leq \tilde{f}(y_0) = |\tilde{f}(x) - \tilde{f}(y_0)| &\leq \sum_{i\in\mathbb{Z}} |\tilde{f}_{B_i} - \tilde{f}_{B_{i+1}}| \\
&\leq C \sum_{i\in\mathbb{Z}} \fint_{2B_i} |\tilde{f} - \tilde{f}_{B_i}|\, d\mu \\
&\leq C \sum_{i\in\mathbb{Z}} 2^{-|i|} d(x,y) \|1+g\|_{L^\infty(\lambda B_i)} \\
&= Cd(x,y) \sum_{i\in\mathbb{Z}} 2^{-|i|} \leq 3Cd(x,y).
\end{aligned}$$

Observe that x is a Lebesgue point of $\tilde{f}$ since $\tilde{f} = 0$ on $B(x,\varepsilon)$. Denote $C' = 3C$. Thus we must have $n \le C'$, with C' depending solely on the doubling constant and the constant of the ∞-Poincaré inequality. Hence if $n > C'$ then the curve family $\Gamma_n = \Gamma(B(x,\varepsilon), B(y,\varepsilon), n)$ must have positive ∞-Modulus, completing the proof in the simple case that $E = B(x,\varepsilon)$ and $F = B(y,\varepsilon)$. The proof for more general E, F is very similar, where we modify the definition of f by looking at curves that connect z to E, and then observing that almost every point in E and almost every point in F are Lebesgue points for the modified function $\tilde{f}$, with $\tilde{f} = 0$ on E and $\tilde{f} \ge nd(x,y)$ on F. This completes the proof of the proposition. □

The following result indicates an advantage of a thick quasiconvex space.

Lemma IV.2.5. *Let X be a thick quasiconvex space. If f is a measurable function (finite μ-a.e.) on X and g is an upper gradient of f, and if B is a ball in X such that $\|g\|_{L^\infty(2CB)} < \infty$, then there is a set $F \subset B$ with $\mu(F) = 0$ such that f is $C\|g\|_{L^\infty(2CB)}$-Lipschitz continuous on $B \setminus F$. Here C is the constant appearing in the definition of thick quasiconvexity.*

Proof. Since f is measurable (and finite μ-almost everywhere), by Lusin's theorem ([42, pp. 61]) for every $n \in \mathbb{N}$ there is a measurable set $E_n \subset X$ such that $\mu(E_n) < 1/n$ and $f_{|B\setminus E_n}$ is continuous. Moreover, for each $n \ge 1$ we can choose G_n be an open set such that $E_n \subset G_n$, $\mu(G_n) < \frac{1}{n}$ (see Theorem 1.10 in [85]) and $f_{|X\setminus G_n}$ is continuous. Now, $V_n = G_1 \cap G_2 \cap \cdots \cap G_n$ is an open set with $\mu(V_n) < \frac{1}{n}$. Observe that $B \setminus V_n = (B \setminus G_1) \cup \cdots \cup (B \setminus G_n)$ and $f_{|B\setminus V_n}$ is continuous.

We will show that f is $C\|g\|_{L^\infty(2CB)}$-Lipschitz continuous on $B \setminus V_n$. Let $P = \{x \in 2CB : g(x) > \|g\|_{L^\infty(2CB)}\}$; then by assumption, $\mu(P) = 0$, and so it follows from Lemma III.2.6 that $\mathrm{Mod}_\infty(\Gamma_P^+) = 0$. To prove that f is $C\|g\|_{L^\infty(2CB)}$-Lipschitz continuous on $B \setminus V_n$, we fix $x, y \in B \setminus V_n$ that are points of density for $B \setminus V_n$. Let $0 < \delta < d(x,y)/4$. By the thick quasiconvexity applied to the sets $E_\delta := B(x,\delta) \setminus V_n$ and $F_\delta := B(y,\delta) \setminus V_n$, there is a curve γ connecting a point $x_\delta \in E_\delta$ and $y_\delta \in F_\delta$ with $\ell(\gamma) \le Cd(x_\delta, y_\delta)$ and $\mathscr{L}^1(\gamma^{-1}(\gamma \cap P)) = 0$. Notice that since x is a point of density for $B \setminus V_n$,

$$\lim_{\rho\to 0} \frac{\mu(B(x,\rho) \cap (B \setminus V_n))}{\mu(B(x,\rho))} = 1,$$

and so $\mu(E_\delta) > 0$. Analogously, we obtain that since y is a point of density for $B \setminus V_n$, $\mu(F_\delta) > 0$. Hence we can apply the thick quasiconvexity property to E_δ and F_δ.

Thus,

$$|f(x_\delta) - f(y_\delta)| \le \int_\gamma g\, ds \le \|g\|_{L^\infty(2CB)}\ell(\gamma) \le C\|g\|_{L^\infty(2CB)} d(x_\delta, y_\delta). \tag{IV.5}$$

Since f is continuous on $B \setminus V_n$, by letting $\delta \to 0$ in (IV.5), we see that

$$|f(x) - f(y)| \leq C\|g\|_{L^\infty(2CB)} d(x,y)$$

as wanted.

Now, we set $F = \bigcap_n V_n$. Note that since $\{V_n\}_n$ is a decreasing sequence of sets, $\mu(F) = \lim_{n\to\infty} \mu(V_n) = 0$. To conclude, let $x, y \in B \setminus F$. Since $B \setminus V_n$ is an increasing sequence of sets, there exists $n \in \mathbb{N}$ such that $x, y \in B \setminus V_n$ and so $f_{|B\setminus F}$ is $C\|g\|_{L^\infty(2CB)}$-Lipschitz. □

In what follows we say that $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ *with comparable energy seminorms* if there is a constant $C > 0$ such that for all $f \in N^{1,\infty}(X)$ there exists $f_0 \in \mathrm{LIP}^\infty(X)$ with $f = f_0$ μ-a.e. and

$$\mathrm{LIP}(f_0) \leq C \inf_g \|g\|_{L^\infty},$$

where the infimum is taken over all ∞-weak upper gradients g of f.

The following example shows that the requirement that $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ as Banach spaces does not by itself imply that these two Banach spaces should have comparable energy seminorms. If however the two seminorms are comparable, then the two Banach space norms are equivalent.

Example IV.2.6. Consider the set $X = \mathbb{R}^2 \setminus \cup_{n=1}^\infty R_n$, where R_n is the open rectangle $R_n = (2n, 2n+1) \times (0, n)$. We endow X with the Euclidean distance and the 2-dimensional Lebesgue measure. It is clear that X is not quasiconvex. Nevertheless, X is uniformly locally thick quasiconvex, that is, for every $p \in X$, the ball $B(p, 1)$ in X with center p and radius 1 is thick quasiconvex with quasiconvexity constant 2. Indeed, if the ball does not contain any corner of the rectangles R_n, $n \in \mathbb{N}$, then it is thick quasiconvex with quasiconvexity constant 1, and if it contains a corner of one of the rectangles R_n then the ball is thick quasiconvex with quasiconvexity constant 2. Now we will see that each $f \in N^{1,\infty}(X)$ coincides a.e. with a function in $\mathrm{LIP}^\infty(X)$. The set $E = \{x \in X : f(x) > \|f\|_{L^\infty}\}$ has measure zero. If $x, y \in X \setminus E$ with $d(x, y) \geq 1/8$, then $|f(x) - f(y)| \leq 2\|f\|_{L^\infty(X)} \leq 16\|f\|_{L^\infty(X)}\, d(x, y)$.

Fix an upper gradient $g \in L^\infty(X)$ of f. Let (p_j) be an enumeration of the points in X having rational coordinates, and for each j consider the ball $B(p_j, 1/2)$. By Lemma IV.2.5, for each j there is a set F_j of measure zero such that f is $2\|g\|_{L^\infty(B(p_j,1))}$-Lipschitz on $B(p_j, 1/2) \setminus F_j$ and hence is $2\|g\|_{L^\infty(X)}$-Lipschitz continuous on $B(p_j, 1/2) \setminus F_j$. The set $F = \cup_{j=1}^\infty F_j \cup E$ is of measure zero. If $x, y \in X \setminus F$ such that $d(x, y) < 1/8$, then there is some j with $x, y \in B(p_j, 1/2)$, and so $|f(x) - f(y)| \leq 2\|g\|_{L^\infty(X)} d(x, y)$. It follows that for all $x, y \in X \setminus F$,

$$|f(x) - f(y)| \leq 2(\|f\|_{L^\infty(X)} + 8\|g\|_{L^\infty(X)})\, d(x, y).$$

Now the restriction $f_{|X\setminus F}$ can be extended to a Lipschitz function on X (for example, via McShane extension, see e.g. [55, Theorem 6.2]). In this way we obtain the equality $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$. Finally, because X is not quasiconvex, it follows from Theorem IV.2.8 below that we do not have comparable energy seminorms for this case.

Proposition IV.2.7. *If X is a thick quasiconvex space, then $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ with comparable energy seminorms.*

Proof. Since we have always that given a Lipschitz function f on X, the constant function $\rho(x) = \mathrm{LIP}(f)$ is an upper gradient of f, we have a continuous embedding $\mathrm{LIP}^\infty(X) \subset N^{1,\infty}(X)$. Hence it suffices to check that we have a continuous embedding $N^{1,\infty}(X) \subset \mathrm{LIP}^\infty(X)$. This follows from Lemma IV.2.5, by exhausting X by balls of large radii and then modifying $f \in N^{1,\infty}(X)$ on the exceptional set of measure zero via McShane extension (see for example [55, Theorem 6.2]). $\square$

We are now ready to state the main result of this chapter. Observe that the following theorem is an improvement of Corollary III.3.5. However, we don't know how to draw the conclusions of Theorem III.3.3 *if the space X just supports a weak ∞-Poincaré inequality.*

Theorem IV.2.8. *Suppose that X is a connected complete metric space supporting a doubling Borel measure μ which is nontrivial and finite on balls. Then the following conditions are equivalent:*

(a) *X supports a weak ∞-Poincaré inequality.*

(b) *X is thick quasiconvex.*

(c) $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ *with comparable energy seminorms.*

(d) *X supports a weak ∞-Poincaré inequality for functions in $N^{1,\infty}(X)$.*

The equivalence of Condition (c) with the other three conditions needs the additional assumption of connectedness of X since the example of the union of two disjoint planar discs satisfies (c) but fails the other three conditions. The other three conditions directly imply that X is connected.

The result (a) $\Longrightarrow$ (d) is immediate, and so the proof of Theorem IV.2.8 is split in three parts:

- (d) $\Longrightarrow$ (b) : has been proven above as Proposition IV.2.4.
- (b) $\Longrightarrow$ (c) : has been proven above as Proposition IV.2.7.
- (c) $\Longrightarrow$ (a) : will be proved in Proposition IV.2.12 below.

Remark IV.2.9. We point out here that if X is complete, connected, and equipped with a nontrivial doubling measure, then the following are equivalent:

(i) X is quasiconvex.

(ii) X supports an ∞-Poincaré inequality for locally Lipschitz continuous functions with continuous upper gradients.

(iii) $\mathrm{LIP}^\infty(X) = D^\infty(X)$ with comparable energy seminorms.

Recall that $\mathrm{LIP}^\infty(X) = D^\infty(X)$ *with comparable energy seminorms* if the two sets are the same and there is a constant $C > 0$ such that for all $f \in \mathrm{LIP}^\infty(X)$,

$$\mathrm{LIP}(f) \le C \sup_{x \in X} \operatorname{Lip} f(x).$$

The implication of (ii)$\Longrightarrow$ (i) is given by the proof of Proposition IV.1.5. We only need to apply the Poincaré inequality to the locally Lipschitz continuous function $\rho_{x,\varepsilon}$ and its continuous upper gradient 1. The implication (i)$\Longrightarrow$ (ii) follows from the argument that if g is a continuous upper gradient of a locally Lipschitz continuous function f, then for $x, y \in X$, by choosing a quasiconvex path γ connecting x to y, we get

$$|f(x) - f(y)| \le \int_\gamma g\, ds \le C\, d(x,y) \sup_{z \in B(x, Cd(x,y))} g(z).$$

So if B is a ball in X and x, y are points in B, then

$$\fint_B \fint_B |f(x) - f(y)|\, d\mu(x)\, d\mu(y) \le C\mathrm{rad}(B) \sup_{z \in CB} g(z) = C\mathrm{rad}(B) \|g\|_{L^\infty(CB)}.$$

The fact that Condition (i) implies Condition (iii) is Corollary II.1.4.

Now suppose that Condition (iii) holds. Then as in the proof of Corollary II.2.8 we conclude that X is quasiconvex. Observe that we are in the hypothesis of Corollary II.2.8 since a complete metric space which supports a doubling measure is proper and therefore locally compact.

Now we continue on to prove Theorem IV.2.8 as outlined before Remark IV.2.9.

The following two technical lemmas will be useful in the sequel.

Lemma IV.2.10. *Suppose $N^{1,\infty}(X) = \mathrm{LIP}^\infty(X)$ with comparable energy seminorms. Then there exists a constant $C \ge 1$ such that for every $E \subset X$ with $\mu(E) = 0$ and for every $x \in X$ and $r > 0$ there is a set $F \subset X$ with $\mu(F) = 0$ so that whenever $y \in X \setminus (B(x, 2r) \cup F)$, there is a rectifiable curve γ_y connecting y to $\overline{B}(x, r)$ such that $\ell(\gamma_y) \le C\, d(x,y)$ and $\mathscr{L}^1(\gamma_y^{-1}(\gamma_y \cap E)) = 0$.*

Proof. Let $E \subset X$ such that $\mu(E) = 0$; since μ is a Borel measure, we may assume (by enlarging E if necessary) that E is a Borel set. Then $\rho = \infty \cdot \chi_E \in L^\infty(X)$ is a nonnegative Borel measurable function. Let Γ_E^+ be the collection of all rectifiable curves γ for which $\mathscr{L}^1(\gamma^{-1}((\gamma \cap E))) > 0$. Then clearly for such curves γ we have $\int_\gamma \rho\, ds = \infty$, and so $\mathrm{Mod}_\infty(\Gamma_E^+) = 0$. As before, we define for $r > 0$,

$$f(z) = \inf_{\gamma \text{ connects } z \text{ to } B(x,r)} \int_\gamma (1+\rho)\, ds,$$

where $\|1+\rho\|_{L^\infty(X)} = 1$. For positive integers k we set $f_k = \min\{k, f\}$. Then $f_k \in N^{1,\infty}(X)$ with $1+\rho$ as an upper gradient (see Lemma IV.1.6), and $f = 0$ on $B(x,r)$. Let F_k be the exceptional set on which f_k has to be modified in order to be Lipschitz continuous; we have $\mu(F_k) = 0$. Observe that since $\mathrm{LIP}^\infty(X) = N^{1,\infty}(X)$ with comparable energy seminorms, there is a constant $C > 0$ such that

$$\mathrm{LIP}(f_k) \le C \inf_g \|g\|_{L^\infty} \le C\|1+\rho\|_{L^\infty(X)} = C,$$

where the infimum is taken over all ∞-weak upper gradients g of f_k.

Let $F = \cup_{k\in\mathbb{N}} F_k$. Thus for $y \in X \setminus (F \cup B(x,2r))$, there exists a positive integer k such that $d(x,y) < k/2C$. In addition,

$$|f_k(y)| = |f_k(y) - f_k(x_1)| \le C\, d(x_1,y) \le C(d(x_1,x) + d(x,y)) \le 2Cd(x,y),$$

for any $x_1 \in B(x,r) \setminus F_k$ and $f_k(y) = f(y)$ is finite. Thus, there exists a rectifiable curve γ_y such that

$$\ell(\gamma_y) + \int_{\gamma_y} \rho\, ds = \int_{\gamma_y} (1+\rho)\, ds \le Cd(x,y).$$

Hence, we have

$$\ell(\gamma_y) \le C\, d(x,y) \qquad \text{and} \qquad \int_{\gamma_y} \rho < +\infty,$$

and so $\mathscr{L}^1(\gamma_y^{-1}(\gamma_y \cap E)) = 0$, as we wanted. $\square$

Lemma IV.2.11. *Let $f \in N^{1,\infty}(X)$ and $g \in L^\infty(X)$ be an upper gradient of f. If $\tilde{f}$ is a Lipschitz continuous function on X such that $f = \tilde{f}$ μ-a.e., then g is an ∞-weak upper gradient of $\tilde{f}$ and so there is a Borel measurable function $0 \le \rho \in L^\infty(X)$ with $\rho = g$ μ-a.e. such that ρ is an upper gradient of $\tilde{f}$.*

Proof. Let $E = \{x \in X : f(x) \ne \tilde{f}(x)\}$; then $\mu(E) = 0$, and so $\mathrm{Mod}_\infty(\Gamma_E^+) = 0$. If $x, y \in X \setminus E$ and β a rectifiable curve connecting x to y in X, then

$$|f(x) - f(y)| = |\tilde{f}(x) - \tilde{f}(y)| \le \int_\beta g\, ds.$$

Let γ be a nonconstant rectifiable compact curve with end points x and y, such that $\gamma \notin \Gamma_E^+$. Then we can find two sequences of points $\{z_i\}$ and $\{w_i\}$ from the trajectory of γ such that for i we have $z_i, w_i \in \gamma \setminus E$ and $z_i \to x$, $w_i \to y$ as $i \to \infty$. Letting γ_i be a subcurve of γ with end points z_i and w_i; then by the above discussion,

$$|\tilde{f}(z_i) - \tilde{f}(w_i)| \le \int_{\gamma_i} g\, ds \le \int_\gamma g\, ds.$$

Since $\tilde{f}$ is Lipschitz continuous, by letting $i \to \infty$ in the above, we get

$$|\tilde{f}(x) - \tilde{f}(y)| \le \int_\gamma g\, ds.$$

It follows that g is an ∞-weak upper gradient of $\tilde{f}$. Since $\mathrm{Mod}_\infty(\Gamma_E^+) = 0$, by Lemma III.2.5, there is a nonnegative Borel measurable function ρ_0 such that $\|\rho_0\|_{L^\infty(X)} = 0$ but for all $\gamma \in \Gamma_E^+$ the integral $\int_\gamma \rho_0\, ds = \infty$. It follows that $\rho = g + \rho_0$ is an upper gradient of $\tilde{f}$ with the desired property. $\square$

Proposition IV.2.12. *Suppose that X is connected and $N^{1,\infty}(X) = \mathrm{LIP}^\infty(X)$ with comparable energy seminorms. Then X supports a weak ∞-Poincaré inequality.*

Proof. Let $f \in N^{1,\infty}(X)$, $g \in L^\infty(X)$ be an upper gradient of f, and fix a ball $B \subset X$. By the assumption that $N^{1,\infty}(X) = \mathrm{LIP}^\infty(X)$ and by Lemma IV.2.11, we may assume that f is itself Lipschitz continuous on X. Let $E = \{w \in 2CB : g(w) > \|g\|_{L^\infty(2CB)}\}$, where C is the constant from Lemma IV.2.10. Then $\mu(E) = 0$. Fix $\varepsilon > 0$.

Observe that since μ is doubling and X is connected, we deduce that $\mu(\{x\}) = 0$ for all $x \in X$ (see condition (I.4)). So for $x \in B$, we can choose $r > 0$ sufficiently small so that

1. $B(x, 2r) \subset B$,
2. $\mu(B(x, 2r)) < \mu(B)/2$,
3. for all $w \in \overline{B}(x, r)$ we have $|f(w) - f(x)| < \varepsilon$ (possible because f is Lipschitz continuous),
4. $\int_{\overline{B}(x,2r)} |f - f(x)|\, d\mu \le \frac{1}{2} \int_B |f - f(x)|\, d\mu$.

Notice that

$$\begin{aligned}\fint_B |f - f(x)| d\mu &= \frac{1}{\mu(B)}\Big(\int_{B(x,2r)} |f - f(x)| d\mu + \int_{B \setminus B(x,2r)} |f - f(x)| d\mu \Big) \\ &\overset{(3.)}{\le} \frac{1}{\mu(B)}\Big(\frac{1}{2} \int_B |f - f(x)| d\mu + \int_{B \setminus B(x,2r)} |f - f(x)| d\mu \Big),\end{aligned}$$

and so
$$\frac{1}{2}\fint_B |f - f(x)|\, d\mu \le \frac{1}{\mu(B)} \int_{B\setminus B(x,2r)} |f - f(x)|\, d\mu. \tag{IV.6}$$

Then,

$$\fint_B |f - f(x)|\, d\mu \overset{\text{(IV.6)}}{\le} \frac{2}{\mu(B)} \int_{B\setminus B(x,2r)} |f - f(x)|\, d\mu \le 2 \fint_{B\setminus B(x,2r)} |f(y) - f(x)|\, d\mu(y).$$

Let $F \subset X$ be the set given by Lemma IV.2.10 with respect to x and r, and for $y \in B \setminus (F \cup B(x,2r))$ let γ_y be the corresponding curve connecting y to $B(x,r)$. We denote the other end point of γ_y as $w_y \in \overline{B}(x,r)$. By the choice of r, we see that $|f(y) - f(x)| \le |f(y) - f(w_y)| + |f(w_y) - f(x)| < |f(y) - f(w_y)| + \varepsilon$. It follows that $|f(y) - f(x)| \le \varepsilon + \int_{\gamma_y} g\, ds \le \varepsilon + C\|g\|_{L^\infty(2CB)} d(x,y)$, where we used the fact that $\mathscr{L}^1(\gamma_y^{-1}(\gamma_y \cap E)) = 0$. Therefore,

$$\begin{aligned}\fint_B |f - f(x)|\, d\mu &\le 2 \fint_{B\setminus(F\cup B(x,2r))} (\varepsilon + C\|g\|_{L^\infty(2CB)} d(x,y)) d\mu(y)\\ &\le 4 \fint_{B\setminus(F\cup B(x,2r))} (\varepsilon + C\|g\|_{L^\infty(2CB)} \mathrm{rad}(B)) d\mu(y)\\ &= 4(\varepsilon + C\|g\|_{L^\infty(2CB)} \mathrm{rad}(B)).\end{aligned}$$

Now integrating over x, we obtain

$$\fint_B \fint_B |f(y) - f(x)|\, d\mu(y)\, d\mu(x) \le 4(\varepsilon + C\|g\|_{L^\infty(2CB)} \mathrm{rad}(B)).$$

Letting $\varepsilon \to 0$ we get the inequality

$$\fint_B \fint_B |f(y) - f(x)|\, d\mu(y)\, d\mu(x) \le 4C\mathrm{rad}(B)\|g\|_{L^\infty(2CB)},$$

which in turn implies, by Remark IV.1.2, the weak ∞-Poincaré inequality for the pair (f, g). Since the constants are independent of f, g, B, we have that (X, d, μ) supports a weak ∞-Poincaré inequality for Newtonian functions. It follows from Proposition IV.2.4 that X is thick quasiconvex.

To complete the proof, we have to check that (X, d, μ) admits a weak ∞-Poincaré inequality for every Borel measurable function $f : X \to \mathbb{R}$ and every upper gradient. Let f be a measurable function and let g be a measurable upper gradient for f. Fix B. If $\|g\|_{L^\infty(2CB)} = \infty$ we are done, so let us assume that $\|g\|_{L^\infty(2CB)} < \infty$. Since by above we have X is thick quasiconvex, we can invoke Lemma IV.2.5 to see that f is Lipschitz in $B \subset X$ up to a set of measure zero. By Lemma IV.2.11, we can assume that f is Lipschitz in all of B and that g is an upper gradient of f in B. Thus we can repeat the proof above for the pair f and g, and the proof is now complete. □

Example IV.2.13. The space (X, d, μ) considered in Example IV.1.4 with a measure that decays very fast to zero at the origin (the point where the two triangular regions are glued) is thick quasiconvex. We can prove it by the aid of Theorem IV.2.8 despite the fact that μ is not doubling. Indeed, since $(X, d, \mathscr{L}^2)$ supports a p-PI for $p > 2$ (see [95, 4.3.1.]), it also supports an ∞-PI. By Theorem IV.2.8, it is also thick quasiconvex (observe that we can apply it since $\mathscr{L}^2$ is a doubling measure). Using the idea in Remark III.2.4, we conclude that (X, d, μ) is also thick quasiconvex.

The rest of this section will be devoted to show that in Theorem IV.2.8 the thick quasiconvexity cannot be replaced with the weaker notion of quasiconvexity.

The next lemma is useful in verifying whether a metric space does not support any Poincaré inequality. Its proof is an adaptation of [21, Lemma 4.3] for the case $p = \infty$.

Lemma IV.2.14. *Let (X, d, μ) be a bounded doubling metric measure space admitting a weak ∞-Poincaré inequality, and let $f : X \longrightarrow I$ be a surjective Lipschitz function from X onto an interval $I \subset \mathbb{R}$. Then, $\mathscr{L}^1_{|I} \ll f_\# \mu$. Here $f_\# \mu$ denotes the push-forward measure of μ under f.*

Proof. Let us denote $L = \mathrm{LIP}(f)$. Suppose the contrary. Then, there exists a Borel set N in I such that $\mathscr{L}^1(N) > 0$ and $\mu(f^{-1}(N)) = f_\# \mu(N) = 0$. On X we consider the function

$$u(x) = \int_0^{f(x)} \chi_N(t) d\mathscr{L}^1(t).$$

This function is L-Lipschitz, because for $x, y \in X$ we have

$$|u(y) - u(x)| = \Big| \int_{f(x)}^{f(y)} \chi_N \, d\mathscr{L}^1 \Big| = \mathscr{L}^1([f(x), f(y)] \cap N) \leq |f(y) - f(x)| \leq L \, d(y, x).$$

Moreover, $g = L \, (\chi_N \circ f)$ is an upper gradient of u. Indeed, for each rectifiable curve $\gamma : [a, b] \longrightarrow X$ one has (without loss of generality we assume that $f(\gamma(a)) < f(\gamma(b))$)

$$|u(\gamma(a)) - u(\gamma(b))| = \Big| \int_{f(\gamma(a))}^{f(\gamma(b))} \chi_N(t) d\mathscr{L}^1(t) \Big| = \mathscr{L}^1([f(\gamma(a)), f(\gamma(b))] \cap N),$$

and

$$\int_\gamma g = \int_a^b L \cdot (\chi_N \circ f(\gamma(t))) d\mathscr{L}^1(t) = L \, \mathscr{L}^1([a, b] \cap (f \circ \gamma)^{-1}(N)).$$

Because γ is arc-length parametrized, $f \circ \gamma$ is L-Lipschitz. It follows that

$$\mathscr{L}^1([a, b] \cap (f \circ \gamma)^{-1}(N)) \geq L^{-1} \, \mathscr{L}^1([f(\gamma(a)), f(\gamma(b))] \cap N),$$

and hence,

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g d\mathscr{L}^1(t)$$

for each rectifiable curve γ in X. However, $\mu\{x \in X : f(x) \in N\} = f_{\#}\mu(N) = 0$ by hypothesis, and so $\chi_N \circ f(x) = 0$ μ-a.e. Therefore by the weak ∞-Poincaré inequality, $\int_X |u - u_X| \, d\mu = 0$, which means that u is constant μ-almost everywhere on X. Because u is Lipschitz continuous on X, it follows that u is constant on X, which contradicts the fact that u is nonconstant on the set $f^{-1}(N)$ (this set is nonempty because f is surjective, and u is not constant here because $\mathscr{L}^1(N) > 0$). $\square$

Example IV.2.15. Let $Q = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\} \subset \mathbb{R}^2$ be the unit square. Divide $R_0 := Q$ into nine equal squares of side length $1/3$ and remove the central one. In this way, we obtain a set R_1 which is the union of 8 squares $Q_{1,j}$ of side length $1/3$. Repeating this procedure on each square we get a sequence of sets R_k, where R_k consists of 8^k squares $Q_{k,j}$ of side length 3^{-k}. We define the *Sierpiński carpet* to be

$$S = S_{\mathbf{3}} = \bigcap_{k \geq 1} R_k.$$

See Figure IV.1.

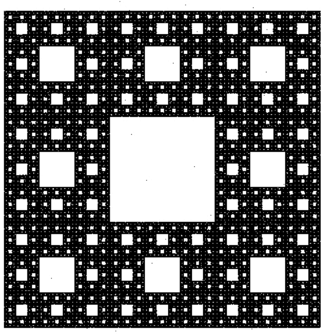


Figure IV.1: Standard Sierpiński carpet $S_{\mathbf{3}}$

If d is the distance in $\mathbb{R}^2$ given by

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|,$$

then (S, d) is a complete geodesic metric space. Let μ be the Hausdorff measure on (S, d) of dimension s, where s is given by the formula, $3^s = 8$. It can be checked that μ is a doubling measure and that the metric d defined above is bi-Lipschitz equivalent to the restriction of the Euclidean metric.

The Sierpiński carpet (S, d, μ) is clearly quasiconvex, and so the following corollary demonstrates that the quasiconvexity property is not sufficient to guarantee ∞-Poincaré inequality.

Corollary IV.2.16. *The Sierpiński carpet (S, d, μ) does not admit a weak ∞-Poincaré inequality.*

Proof. Let f be the projection on the horizontal axis. It can be checked that $f_{\#}\mu \perp \mathscr{L}^1$ (see [21, 4.5]). Indeed, as shown in [21], given a point $0 < x < 1$, by the way of ternary expansion of x we can see that the interval I_n centered at x of radius 3^{-n} has Lebesgue measure $\mathscr{L}^1(I_n) \approx 3^{-n}$, but $f_{\#}\mu(I_n) \approx \exp(-\psi(x,n))$ for appropriately chosen function ψ, with the property that

$$\lim_{n\to\infty} \frac{f_{\#}\mu(I_n)}{\mathscr{L}^1(I_n)} \approx \limsup_{n\to\infty} \frac{\exp(-\psi(x,n))}{3^{-n}}$$

which is for $\mathscr{L}^1$-a.e. x either 0 or ∞. This fact, in conjunction with the Radon-Nikodym Theorem, implies that $f_{\#}\mu$ is singular with respect to the Lebesgue measure $\mathscr{L}^1$.

The result now follows from Lemma IV.2.14. $\square$

IV.3 p-Poincaré inequality vs. ∞-Poincaré inequality

In what follows, we study a concept analogous to thick quasiconvexity associated with p-Poincaré inequality for finite $p \geq 1$, *p-thick quasiconvexity* . It is known that if a complete doubling metric measure space supports a weak p-Poincaré inequality, then the space is *quasiconvex* , that is, there exists a constant such that every pair of points can be connected with a curve whose length is at most the constant times the distance between the points (see [93] or [59]). See [79] for further improvements of the quasiconvexity condition. In what follows, we consider a stronger geometric property. We will prove that every pair of sets of positive measure which are a positive distance apart can be connected by a "thick" family of quasiconvex curves in the sense that the modulus of this family of curves is positive. The following definition makes this idea more precise.

Definition IV.3.1. A metric measure space (X, d, μ) is said to be a *p-thick quasiconvex* space (where $1 \leq p \leq \infty$) if there is a constant $C \geq 1$ such that for all $x, y \in X$, all $0 < \varepsilon < \frac{1}{4}d(x,y)$, and all measurable sets $E \subset B(x,\varepsilon)$ and $F \subset B(y,\varepsilon)$ satisfying $\mu(E)\mu(F) > 0$, we have that

$$\mathrm{Mod}_p(\Gamma(E,F,C)) > 0.$$

Here $\Gamma(E,F,C)$ denotes the set of all curves $\gamma_{p,q}$ connecting $p \in E$ and $q \in F$ with $\ell(\gamma_{p,q}) \leq Cd(p,q)$. Recall here that we say that X is *thick quasiconvex* if it is ∞-thick quasiconvex.

Remark IV.3.2. Note that every complete p-thick quasiconvex space X supporting a doubling measure is quasiconvex. For a proof, see Remark IV.2.2. The converse is not true in general. The Sierpiński carpet is a quasiconvex space which is not ∞-thick quasiconvex (see Corollary IV.2.16), and so it is not p-thick quasiconvex either for any $1 \leq p \leq \infty$.

Lemma IV.3.3. *Whenever $1 \leq p \leq \infty$ the Euclidean space $\mathbb{R}^n$ is p-thick quasiconvex with quasiconvexity constant $C = 1$.*

Proof. An easy modification of Lemma IV.3.7 tells us that it suffices to prove that $\mathbb{R}^n$ is p-thick quasiconvex for $p = 1$.

Let $x, y \in \mathbb{R}^n$ be two distinct points, and $0 < \varepsilon < |x - y|/10$. Let $E \subset B(x, \varepsilon)$ and $F \subset B(y, \varepsilon)$ be two measurable sets of positive measure, and $\Gamma(E, F, 1)$ be the collection of all straight line segments connecting points in E to points in F. We wish to show that $\mathrm{Mod}_1(\Gamma(E, F, 1)) > 0$. To do this, let $z \in E$ be a point of density 1 of E, and $w \in F$ be a point of density 1 of F; since both E and F have positive measure, by Lebesgue differentiation theorem such points exist. Let L be the line passing through z and w, P_1 be the $(n-1)$-dimensional hyperplane perpendicular to L, and P_2 the $(n-1)$-dimensional hyperplane parallel to P_1, such that the balls $B(z, 2\varepsilon)$ and $B(w, 2\varepsilon)$ lie between these two hyperplanes.

Let E_1 be the orthogonal projection of E to P_1 and F_1 the orthogonal projection of F to P_2. By Fubini's theorem, we know that $\mathcal{H}^{n-1}(E_1) > 0$ and $\mathcal{H}^{n-1}(F_1) > 0$, and that the projection z_1 of z to P_1 is a point of $\mathcal{H}^{n-1}$-density 1 for E_1 and the projection w_1 of w to P_2 is a point of $\mathcal{H}^{n-1}$-density 1 for F_1. Let Γ be the collection of all line segments *parallel to* L and connecting points in E_1 to points in F_1. We now show that $\mathrm{Mod}_1(\Gamma) > 0$. Suppose $\mathrm{Mod}_1(\Gamma) = 0$. Then lines parallel to L and passing through $\mathcal{H}^{n-1}$-almost every point in E_1 does not intersect F_1, and lines parallel to L and passing through $\mathcal{H}^{n-1}$-almost every point in F_1 does not intersect E_1 (see the discussion of [102, Chapter 1, Section 7.2]). Let $\nu_1 = \chi_{E_1}\mathcal{H}^{n-1}$, and $\nu_2 = \chi_{F_1}\mathcal{H}^{n-1}$. It follows that the projection ν_2' of the measure ν_2 to the hyperplane P_1 is mutually singular with ν_1. However, because w_1 is a $\mathcal{H}^{n-1}$-point of density 1 for F_1, and the projection of w_1 to P_1 is the same as z_1, it follows that z_1 is a point of density 1 for both ν_1 and the projection ν_2' of ν_2 to P_1; this is not possible since by the mutual singularity of the two measures, and by the definition of the two measures, for $r > 0$,

$$\nu_2'(B(z_1, r)) + \nu_1(B(z_1, r)) \leq \mathcal{H}^{n-1}|_{P_1}(B(z_1, r)).$$

It follows that $\mathrm{Mod}_1(\Gamma) > 0$. Since every curve in Γ has a subcurve in $\Gamma(E, F, 1)$, we obtain that $\mathrm{Mod}_1(\Gamma(E, F, 1) \geq \mathrm{Mod}_1(\Gamma) > 0$, which concludes the proof. ☐

Remark IV.3.4. The proof of the above lemma also tells us that in the Euclidean setting, given two parallel $(n-1)$-dimensional hyperplanes of $\mathbb{R}^n$ and two sets of $\mathcal{H}^{n-1}$-measure positive, one lying in one of the hyperplanes and the other lying in the other hyperplane, the set of all geodesic line segments connecting points in one set to points in the second set has positive 1-modulus.

The following proposition gives an analog of Condition (a) implying Condition (b) in Theorem IV.2.8 for the case of finite p; the converse is not true in general, as will be shown below.

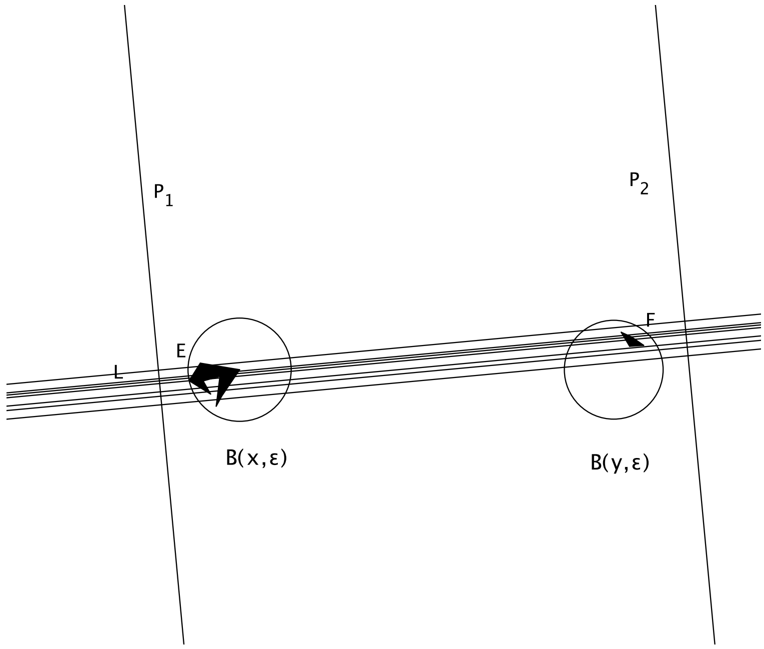


Figure IV.2: Thick quasiconvex family

Proposition IV.3.5. *Let (X, d, μ) be a metric measure space with μ a doubling measure. If X supports a weak p-Poincaré inequality for functions in $N^{1,p}(X)$ with upper gradients in $L^p(X)$, then X is p-thick quasiconvex.*

Proof. Let $x, y \in X$ such that $x \neq y$, and let $0 < \varepsilon < d(x,y)/4$. Fix $n \in \mathbb{N}$ and let $\Gamma_n = \Gamma(B(x,\varepsilon), B(y,\varepsilon), n)$ be the collection of all rectifiable curves connecting $B(x,\varepsilon)$ to $B(y,\varepsilon)$ such that $\ell(\gamma) \leq n\, d(x,y)$. By the choice of ε, if p, q are the end points of γ, then $d(p,q)/4 \leq d(x,y) \leq 4d(p,q)$.

Suppose that $\operatorname{Mod}_p(\Gamma_n) = 0$. Then, there exists a nonnegative Borel measurable function $g \in L^p(X)$ such that $\|g\|_{L^p(X)} = 1$ and for all $\gamma \in \Gamma_n$, the path integral $\int_\gamma g\, ds = \infty$.

Next, for each $k \geq 1$ consider the family of functions $g_k = \frac{1}{k}\, g$. It is clear that $\|g_k\|_{L^p(X)} = 1/k$. By the Maximal Function Theorem (see [55, 2.2]),

$$\mu(\{z \in X : M(g_k^p)(z) > 1\}) \leq \frac{C}{1} \int_X g_k^p = C\|g_k\|^p_{L^p(X)} < C\frac{1}{k^p}. \tag{IV.7}$$

Recall here that $M(g)(z) = \sup_{r>0} \fint_{B(z,r)} |g| d\mu$.

Let

$$S_k = \{z \in X : M(g_k^p)(z) \leq 1\} = \{z \in X : M(g^p)(z) \leq k^p\}.$$

Observe that if $k_1 \leq k_2$ then $S_{k_1} \subset S_{k_2}$. Moreover, by inequality (IV.7), the set $G =$

$X \setminus \bigcup_{k\geq 1} S_k$ has measure zero . Let

$$u_k(z) = \inf_{\gamma \text{ connecting } z \text{ to } B(x,\varepsilon)} \int_\gamma (1+g_k)\, ds.$$

Note that $u_k = 0$ on $B(x,\varepsilon)$ for each k. If $z \in B(y,\varepsilon)$ and γ is a rectifiable curve connecting z to $B(x,\varepsilon)$, then either $\gamma \in \Gamma_n$ in which case $\int_\gamma (1+g_k)\, ds \geq \int_\gamma g_k\, ds = \infty$, or else $\gamma \notin \Gamma_n$, in which case $\ell(\gamma) > n\, d(x,y)$ and so $\int_\gamma (1+g_k)\, ds \geq \int_\gamma 1\, ds > n\, d(x,y)$. Hence $u_k(z) \geq n\, d(x,y)$. It follows that the function $v_k = \min\{u_k, n\, d(x,y)\}$ has the properties that

1. $v_k = 0$ on $B(x,\varepsilon)$,
2. $v_k = n\, d(x,y)$ on $B(y,\varepsilon)$,
3. $1+g_k$ is an upper gradient of v_k on X,
4. $v_k \in N^{1,p}(X)$.

Since $\mu(G) = 0$ we can find points $x_0 \in B(x,\varepsilon/4) \setminus G$ and $y_0 \in B(y,\varepsilon/4) \setminus G$. Let $k \in \mathbb{N}$ such that $x_0, y_0 \in S_k$. By using the chain of balls $B_i = B(x_0, 2^{1-i} d(x,y))$ if $i \geq 0$ and $B_i = B(y_0, 2^{1+i} d(x,y))$ if $i \leq -1$, and by the weak p-Poincaré inequality,

$$\begin{aligned}
n\, d(x,y) = v_k(y_0) = |v_k(x_0) - v_k(y_0)| &\leq \sum_{i\in\mathbb{Z}} |v_{kB_i} - v_{kB_{i+1}}| \\
&\leq C \sum_{i\in\mathbb{Z}} \fint_{2B_i} |v_k - v_{kB_i}|\, d\mu \\
&\leq C \sum_{i\in\mathbb{Z}} 2^{-|i|} d(x,y) \Big(\frac{1}{\mu(\lambda B_i)} \int_{\lambda B_i} (1+g_k)^p\, d\mu \Big)^{1/p} \\
&\leq C \sum_{i\in\mathbb{Z}} 2^{-|i|} d(x,y) \Big(1 + \Big(\frac{1}{\mu(\lambda B_i)} \int_{\lambda B_i} (g_k)^p\, d\mu \Big)^{1/p} \Big) \\
&\leq C d(x,y) \sum_{i\in\mathbb{Z}} 2^{-|i|} (1+1) \leq C d(x,y).
\end{aligned}$$

Observe that x_0, y_0 are Lebesgue points of v_k since $v_k = 0$ on the open set $B(x,\varepsilon) \ni x_0$ and $v_k = n\, d(x,y)$ on $B(y,\varepsilon) \ni y_0$. Thus we must have $n \leq C$, with C depending solely on the doubling constant and the constant of the p-Poincaré inequality. Hence if $n > C$ then the curve family $\Gamma_n = \Gamma(B(x,\varepsilon), B(y,\varepsilon), n)$ must have positive p-Modulus, completing the proof in the simple case that $E = B(x,\varepsilon)$ and $F = B(y,\varepsilon)$. The proof for more general E, F is very similar, where we modify the definition of u_k by looking only at curves that connect z to E, and then observing that almost every point in E and almost every point in F are Lebesgue points for the modified function v_k, with $v_k = 0$ on E and $v_k = nd(x,y)$ on F. This completes the proof of the proposition. □

As we have seen in Theorem IV.2.8, the converse of Proposition IV.3.5 for the case $p = \infty$ is true. The following example shows that for finite p, the converse of Proposition IV.3.5 is not true in general. The metric measure space in this example, being thick quasiconvex and hence supporting a weak ∞-Poincaré inequality, also demonstrates that there is no self-improvement of ∞-Poincaré inequality in the spirit of Keith-Zhong [73].

Example IV.3.6. Let $Q = [0,1] \times [0,1] \subset \mathbb{R}^2$ be the unit square. Let Q_1 be the set obtained by dividing Q into nine equal squares of side-length $1/3$ and removing the central *open* square. The set Q_1 is the union of 8 squares of side-length $1/3$. Repeating this procedure on each of the 8 squares making up Q_1 we obtain the set Q_2, a union of 8^2 squares, each of side-length $1/3^2$. Repeating this process we get a sequence of sets Q_j consisting of 8^j squares of side-length $1/3^j$. Notice that each Q_j has positive area, so we can define a probability measure μ_j concentrated on Q_j obtained by renormalizing the Lebesgue measure (restricted to Q_j) to have measure one. The metric measure space under consideration is

$$X = Q_1 \cup (Q_2 + (1,0)) \cup (Q_3 + (2,0)) \cup \cdots (Q_j + (j-1,0)) \cup \cdots$$

endowed with the measure

$$\mu = \sum_i \chi_{Q_j + (j-1,0)} \cdot \mu_j,$$

and with the Euclidean metric restricted to X. Here, $Q_j + (j-1,0)$ is the set obtained by translating Q_j in the direction parallel to the x-axis by $j-1$ units;

$$Q_j + (j-1,0) := \{(x+j-1, y) \in \mathbb{R}^2 \,:\, (x,y) \in Q_j\},$$

and μ_j is the measure given by

$$\mu_j = (9/8)^j \, \mathscr{L}^2|_{Q_j + (j-1,0)}.$$

It can be directly verified that the measure μ is doubling on X.

Suppose that (X, d, μ) supports a weak p-Poincaré inequality for some finite p with constants C_p and λ. By [12, Theorem 4.4], uniform domains in (X, d, μ) also support a weak p-Poincaré inequality with constants C_p' and λ', where C_p' and λ' depend solely on C_p, λ, and the uniformity constant of the uniform domain. Recall here that a domain $\Omega \subset X$ is C-*uniform*, $C \geq 1$ if for every pair of points $x, y \in \Omega$ there is a C-quasiconvex curve γ in Ω connecting x and y and for all $z \in \gamma$,

$$\min\{\ell(\gamma_{xz}), \ell(\gamma_{yz})\} \geq C\, d(z, X \setminus \Omega).$$

Here γ_{xz} and γ_{yz} are subcurves connecting z to x and y, respectively. For each j, the domains $Q_j + (j-1)$ are uniform domains in X, with the same uniformity constant. To see this, note that the unit square $Q = [0,1] \times [0,1]$ is a C_0-uniform domain in $\mathbb{R}^2$. Let

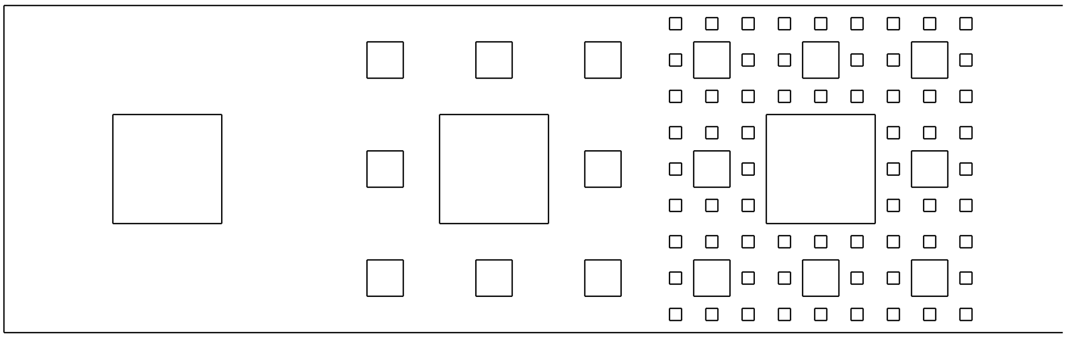


Figure IV.3: Sierpiński strip

$P_1 = (x_1, y_1), P_2 = (x_2, y_2) \in Q_j + (j-1, 0)$; then $(x_1 - j + 1, y_1), (x_2 - j + 1, y_2) \in Q$, and so there is a C_0-uniform curve β in Q connecting these two points. The curve $\beta_j := \beta + (j-1, 0)$ may not lie in $Q_j + (j-1, 0)$, but if it does not, then it intersects the translations of squares removed from Q in order to obtain Q_j; these removed squares are of Whitney type in Q (because the distance from the removed square to the boundary of Q is at least 1/3 the side-length of the removed square), and so modifying the path β in the following manner yields a curve γ connecting P_1 to P_2 in $Q_j + (j-1, 0)$ which is $\sqrt{2}\, C_0$-uniform. If β_j intersects the translation of one of the removed open squares, then with P_1', P_2' denoting the points of intersection of β_j with the boundary of the removed square, we may replace the sub-curve of β_j inside this removed square with the shortest of the two components obtained by removing P_1', P_2' from the boundary of the removed square. This sub-curve replacing the original sub-curve of β_j has length no more than $\sqrt{2}$ times the length of the sub-curve being replaced; furthermore, its distance from the boundary (with respect to X) of $Q_j + (j-1, 0)$ is comparable to the corresponding quantity of the original sub-curve, with comparison constant $\sqrt{2}$. Hence the curve γ obtained by modifying β_j as above results in a uniform curve in $Q_j + (j-1, 0)$ with uniformity constant $\sqrt{2}\, C_0$, which is independent of j. One should keep in mind here that the boundary of $Q_j + (j-1, 0)$ in X is the union of the two vertical line segments $\{j-1\} \times [0, 1]$ and $\{j\} \times [0, 1]$, whose translation by $(-j+1, 0)$ is a subset of the boundary of Q in $\mathbb{R}^2$.

As explained above, the domains $Q_j + (j-1)$ are uniform domains in X, with the same uniformity constant. Therefore, for each $j \geq 1$ the space $(Q_j + (j-1, 0), d, \mu_j)$ supports a weak p-Poincaré inequality for some finite p with constants C_p' and λ' and so it is clear that the space (Q_j, d_j, μ_j), where d_j is the Euclidean distance restricted to each Q_j, also supports a weak p-Poincaré inequality with the same constants C_p' and λ'.

The sequence of pointed spaces $\{(Q_j + (j-1, 0), d_j, \mu_j, (j-1, 0))\}$ converges in the measured Gromov-Hausdorff topology to the space (S, d, μ), where μ is the weak* limit of the probability measures μ_j and $S = \bigcap Q_j$ is the *Sierpiński carpet.* By the construction of the carpet, it is easy to see that the sequence of compact subsets $\{Q_j\}_j$ of $\mathbb{R}^2$ converges in the Hausdorff topology to the Sierpiński carpet equipped with the Euclidean metric; hence the convergence of $\{(Q_j + (j-1,0), d_j, (j-1,0))\}$ holds in the Gromov-Hausdorff sense as well. The sequence of measures μ_j converges to an Ahlfors s-regular measure, where the number s is given by $3^s = 8$. In particular, μ coincides with the Hausdorff measure on (S,d) of dimension s, see [41, Page 130, Theorem 9.3] together with [85, Theorem 1.23] or [94, Section 4.1]. Furthermore, μ is a doubling measure, and in fact is an Ahlfors regular measure on the carpet S.

Since p-Poincaré inequalities (with uniformly bounded constants) persists through the limit of a sequence of converging pointed metric measure spaces (see [24, Theorem 9.6] or [69, Theorem 3]), the limit space (S,d,μ) would support a weak p-Poincaré inequality, which is known to be not true. See for example [21, Prop. 4.5] or [94].

However, it is clear that (X,d,μ) is p-thick quasiconvex. We can use a simple modification of the proof of Lemma IV.3.3 to families of curves, obtained as a union of line segments parallel to the two coordinate axes. These curves are at most 2-quasiconvex. The idea is that for any pair of points $x,y\in X$, one can find a narrow 2-quasiconvex tube of curves connecting balls centered at x and y of positive p-modulus. If the largest index j for which one of x,y lies in $Q_j + (j-1,0)$ is large, then this tube of curves is correspondingly narrow and has a small p-modulus. Thus the p-modulus of curves connecting the balls has no quantitative lower bound, and that is the reason why the space does not support a weak p-Poincaré inequality for any finite p.

As we have seen before, p-thick quasiconvexity does not guarantee a weak p-Poincaré inequality for finite p. However, the next lemma shows that it is a sufficient condition to obtain a weak ∞-Poincaré inequality.

Lemma IV.3.7. *If (X,d,μ) is a p-thick quasiconvex for some $p<\infty$ then (X,d,μ) supports a weak ∞-Poincaré inequality.*

Proof. Since (X,d,μ) is a p-thick quasiconvex space, we know that there exists $C\geq 1$ such that for all $x,y\in X$, $0<\varepsilon<\frac{1}{4}d(x,y)$, and all measurable sets $E\subset B(x,\varepsilon)$, $F\subset B(y,\varepsilon)$ satisfying $\mu(E)\mu(F)>0$ we have that $\mathrm{Mod}_p(\Gamma(E,F,C))>0$, where $\Gamma(E,F,C)$ denotes the set of curves $\gamma_{p,q}$ connecting $p\in E$ and $q\in F$ with $\ell(\gamma_{p,q})\leq Cd(p,q)$. Let g be a nonnegative Borel measurable function on X such that for all $\gamma\in\Gamma(E,F,C)$ we have $\int_\gamma g\,ds\geq 1$. Since the curves $\gamma\in\Gamma(E,F,C)$ are of length at most $4C\,d(x,y)$ and hence lie in the ball $B:=B(x,8C\,d(x,y))$, and so we may assume, without loss of generality, that the support of g lies in B. Because $0<\mu(B)<\infty$ we obtain by Hölder's inequality that

$$\|g\|_{L^p(X)}\leq \mu(B)^{\frac{1}{p}-\frac{1}{s}}\|g\|_{L^s(X)} \qquad \text{for all } s\in(p,\infty).$$

Letting $s\to\infty$ we get that $\|g\|_{L^p(X)}\leq\mu(B)^{\frac{1}{p}}\|g\|_{L^\infty(X)}$, and so

$$\big(\mathrm{Mod}_p(\Gamma(E,F,C))\big)^{\frac{1}{p}}\leq\mu(B)^{\frac{1}{p}}\,\mathrm{Mod}_\infty(\Gamma(E,F,C)).$$

The last inequality says that if $\mathrm{Mod}_p(\Gamma(E,F,C)) > 0$, then $\mathrm{Mod}_\infty(\Gamma(E,F,C)) > 0$ and so (X,d,μ) is a thick quasiconvex space. By the geometric characterization in Theorem IV.2.8, we conclude that (X,d,μ) supports a weak ∞-Poincaré inequality. □

Remark IV.3.8. By the aid of Lemma IV.3.7, the space (X,d,μ) in Example IV.3.6 is a doubling metric measure space which supports a weak ∞-Poincaré inequality but does not support a weak p-Poincaré inequality for any finite p. Observe that in Example IV.1.4 we have already constructed a space which supports a weak ∞-Poincaré inequality but does not support a weak p-Poincaré inequality for any finite p. However, the measure considered in that example was not doubling.

Finally in this section, we give an example which shows that the weak ∞-Poincaré inequality does not persist under Gromov-Hausdorff convergence.

Example IV.3.9. We consider the sets Q_j constructed in Example IV.3.6 above, and the corresponding Hausdorff limit of $\{Q_j\}_j$, which is the Sierpiński carpet $S = \bigcap Q_j$. The sequence of metric measure spaces under consideration is $\{(Q_j, d_j, \mu_j)\}$, where d_j is the Euclidean distance restricted to each Q_j and μ_j is a probability measure concentrated on Q_j. As mentioned in Example IV.3.6, the sequence of pointed spaces $\{(Q_j, d_j, \mu_j, (0,0))\}$ converges to the space $(S, d, \mu, (0,0))$, where μ is the Hausdorff measure corresponding to the Hausdorff dimension of S. The metric measure spaces in the sequence $\{(Q_j, d_j, \mu_j)\}$ are thick quasiconvex (and therefore support a weak ∞-Poincaré inequality). Keep in mind that in this example the constant that appears in the ∞-thick quasiconvexity property for each Q_j depends only on the constant of the quasiconvexity of the space. Therefore the constants are uniformly bounded by the quasiconvexity constant. However, the limit space (S,d,μ) does not support a weak ∞-Poincaré inequality (see Corollary IV.2.16).

IV.4 Open problems

We conclude the chapter by listing some problems in this section, for which we have so far neither a counterexample nor a proof.

Cheeger proved in [24] that doubling p-Poincaré spaces admit a differentiable structure for which Lipschitz functions are differentiable μ-a.e. A remarkable fact is that although the exponent p is present in the hypothesis of this result, it has no role in the conclusions. Keith in [69] (see also [70]) weakened the hypotheses so as not depend on p. He defined the $\mathrm{Lip}-\mathrm{lip}$ condition as follows: A metric measure space X is said to satisfy a $\mathrm{Lip}-\mathrm{lip}$ *condition* if there exists a constant $K \geq 1$ such that

$$\mathrm{Lip}\, f(x) \leq K\, \mathrm{lip}\, f(x)$$

for all Lipschitz functions $f : X \longrightarrow \mathbb{R}$, for μ-a.e. $x \in X$ (the exceptional set of measure zero is of course allowed to depend on f). Here $\mathrm{lip}\, f$ is defined as $\mathrm{Lip}\, f$ changing $\limsup$ by lim inf. As a consequence of [24, Theorem 6.1] and the fact that $\operatorname{lip} f$ is also a weak upper gradient of any Lipschitz function f, we know that complete doubling metric measure spaces which admit a weak p-Poincaré inequality satisfy the $\operatorname{Lip} - \operatorname{lip}$ condition as well. The thesis [70, Section 1.4] conjectures that this generalization can be understood as a version of Cheeger's theorem for $p = \infty$. The following example shows that these two conditions are not equivalent.

Example IV.4.1. Let $X \subset \mathbb{R}^2$ be the set obtained by removing certain thin rectangles from $[0,1] \times [0,1]$ as follows:

$$X = [0,1] \times [0,1] \setminus \bigcup_{2 \le n \in \mathbb{N}} \left(\frac{1}{n} - \frac{1}{n^4}, \frac{1}{n} \right) \times \left(\frac{1}{2}, 1 \right).$$

We consider the complete space (X, d, μ) where d is the Euclidean distance and μ is the 2-dimensional Lebesgue measure $\mathscr{L}^2$ restricted to X. The space is not quasiconvex and so it cannot support any weak p-Poincaré inequality for $1 \le p \le \infty$. However, since it is an open set of $\mathbb{R}^2$ (except for the boundary, which has zero $\mathscr{L}^2$-dimensional measure), it satisfies the $\operatorname{Lip} - \operatorname{lip}$ condition. Furthermore, it can be checked that since the rectangles $(n^{-1} - n^{-4}, n^{-1}) \times (2^{-1}, 1)$ removed from $[0,1] \times [0,1]$ are sufficiently thin, the measure on X is doubling.

As we have seen in the Introduction, some of the classical theorems in analysis in the Euclidean setting can be extended to doubling metric measure spaces. The Lebesgue Differentiation Theorem is such an example: if f is a locally integrable function on a doubling metric space X, then

$$f(x) = \lim_{r \to 0} \left(\fint_{B(x,r)} f^p \, d\mu \right)^{1/p},$$

for μ-a.e. point in X. In other words, almost every point in X is a *Lebesgue point* for f, see for example [55, Theorem 1.8]. One of the difficulties when working with the L^∞-norm is that the Lebesgue differentiation Theorem is no longer true. That is, there are examples for which

$$f(x) \neq \lim_{r \to 0} \|f\|_{L^\infty(B(x,r))},$$

in a set of positive measure. This fact makes proving that a weak ∞-Poincaré inequality implies a $\operatorname{Lip} - \operatorname{lip}$ condition a difficult task.

Question 1: Is it true that when a complete doubling metric measure space supports a weak ∞-Poincaré inequality, it must necessarily satisfy a $\operatorname{Lip} - \operatorname{lip}$ condition? Even if such a space does not satisfy a $\operatorname{Lip} - \operatorname{lip}$ condition, does it support a nontrivial (that is, there is a Lipschitz function whose derivative is nonvanishing on a set of positive measure) measurable differentiable structure in the sense of [24],[70]? We point out here that by the results in [70], the $\operatorname{Lip} - \operatorname{lip}$ condition together with the doubling measure by itself guarantees a measurable differentiable structure, but this structure may not be natural in the sense that there may be a Lipschitz function whose derivative vanishes on an open connected set without the function itself being constant on that open connected set. If the metric space satisfies a weak ∞-Poincaré inequality in addition to doubling and a $\mathrm{Lip} - \mathrm{lip}$ condition, then the Poincaré inequality forces the function, whose derivative vanishes on a connected open set, to be itself constant on that connected open set.

Question 2: There exist metric measure spaces that are ∞-thick quasiconvex but are not p-thick quasiconvex for any finite $p \geq 1$ (see Example IV.1.4); however, the examples we know of, are not doubing measure spaces. We have already seen that there are doubling metric measure spaces that are ∞-thick quasiconvex and hence support an ∞-Poincaré inequality but fail to support a weak p-Poincaré inequality for any finite p; however, these examples are p-thick quasiconvex for some finite p. Are there doubling ∞-thick quasiconvex spaces which are not p-thick quasiconvex?

Chapter V

Rectifiable curves in Sierpiński carpets

From the previous sections it has become clear that, to obtain a setting where the type of calculus we are looking at is possible, we need a space which not only has rectifiable curves, but also plenty of them uniformly at all scales. It is known that some classical fractals, such as the Sierpiński carpets (and Sierpiński Gaskets) have rectifiable curves (they are indeed quasiconvex), but they are not enough for our purposes; that is, in terms of modulus and Poincaré inequalities (see Example IV.2.16 and discussion in [94, 2.3]).

On the other hand, in the last years fractal geometry has developed quickly on the foundation of geometric measure theory, harmonic analysis, dynamical systems and ergodic theory. For example, one can construct an analogous operator to the Laplacian on fractals in order to deal with continuous transport problems like heat conduction (see [101] and references therein). Brownian motion on the Sierpiński carpet has also attracted interest in recent years [9].

A *carpet* is a metric space which is homeomorphic to $S_{\mathbf{3}}$ (see definition in IV.2.15). The following fundamental problem arises in the study of quasiconformal and bi-Lipschitz maps between carpets: characterize the rectifiable curves contained in a given carpet.

For instance, such a characterization could perhaps be used to give a direct proof of the following bi-Lipschitz rigidity property of $S_{\mathbf{3}}$: *every bi-Lipschitz map of $S_{\mathbf{3}}$ onto itself is the restriction of an isometry of the plane which preserves the unit square Q.* The bi-Lipschitz rigidity of $S_{\mathbf{3}}$ is a corollary of the quasisymmetric rigidity, which has been established by Bonk and Merenkov [20] using conformal modulus techniques. As far as we are aware, there is no independent proof of bi-Lipschitz rigidity which does not use conformal methods. Further results on the conformal geometry of carpets can be found in [71], [19], [17], [86], [83]. We remark that the conformal geometry of carpets arises in connection with the Kapovich–Kleiner conjecture on quasisymmetric uniformization of Sierpiński carpet group boundaries. See [18] for additional details.

Let us consider planar carpets, i.e., carpets which are realized as subsets of the plane. Every rectifiable curve contained in such a carpet is, in particular, a rectifiable curve in the plane and hence admits a tangent line at almost every point by the theorem of Rademacher [90], [84, Theorem 7.3]. We thus naturally begin by considering the line segments contained in such carpets. Our starting point is the following folklore observation: *there exist points* *in S_3 which are joined by straight line segments which lie entirely within S_3, yet are not horizontal or vertical.* See Figure V.1 for an illustration of some of these line segments.

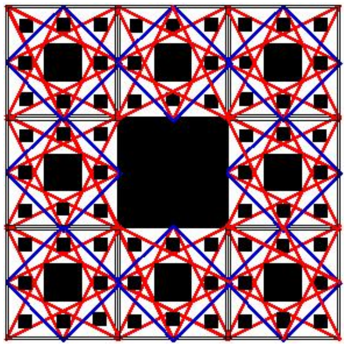


Figure V.1: Line segments contained in S_3

From the figure, we see that the set of slopes of nontrivial line segments contained in S_3 is

$$\left\{0, \pm\frac{1}{2}, \pm 1, \pm 2, \infty\right\}.$$

A proof of this fact was given by Bandt and Mubarak [8].

A planar carpet S is called a *square carpet* if the bounded components of $\mathbb{R}^2 \setminus S$ are Euclidean squares. The boundaries of the omitted square domains are called the *peripheral squares* of S.

Our aim is to give a complete description of the slopes of nontrivial line segments contained in the members of a class of square Sierpiński carpets. In Section V.1 we introduce the class of carpets under consideration. In Section V.2 we will characterize the slopes of nontrivial line segments contained in self-similar Sierpiński carpets (Theorem V.2.1 and Theorem V.2.10). In addition, the set of slopes is related to Farey sequences and the dynamics of punctured square toral billiards. As a consequence, we deduce in Section V.3 conclusions about the collection of everywhere differentiable curves contained in such carpets. These results provide a first step towards a description of the rectifiable curves contained in such carpets.

V.1 Self-similar and nonself-similar Sierpiński carpets

Let

$$\mathbf{a} = (a_1^{-1}, a_2^{-1}, \dots) \in \left\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\right\}^{\mathbb{N}}.$$

Divide $R_0 := Q$ into a_1^2 equal squares of side length a_1^{-1} and remove the central one. We obtain a set R_1 which is the union of $a_1^2 - 1$ squares $Q_{1,j}$ of side length a_1^{-1}. Now, consider the remaining $a_1^2 - 1$ squares and divide each of them into a_2^2 squares of side length $a_1^{-1} \cdot a_2^{-1}$ and remove each open central square again. Iterating this procedure we get a sequence of sets R_k, where R_k consists of

$$(a_1^2 - 1) \cdot (a_2^2 - 1) \ldots (a_j^2 - 1)$$

squares $Q_{k,j}$ of side length $a_1^{-1} \cdot a_2^{-1} \ldots a_k^{-1}$. We define the *generalized Sierpiński carpet* to be

$$S_{\mathbf{a}} = \bigcap_{k \geq 1} R_k.$$

For any sequence $\mathbf{a}$, the carpet $S_{\mathbf{a}}$ is a compact set without interior which is rectifiably connected. The set $S_{\mathbf{a}}$ has Hausdorff dimension two if and only if the sequence $\mathbf{a}$ is in c_0, i.e., $a_j^{-1} \to 0$. Furthermore, $S_{\mathbf{a}}$ has positive area (Lebesgue 2-measure) if and only if $\mathbf{a} \in \ell^{\mathbf{2}}$, i.e., $\sum_j a_j^{-2} < \infty$. The metric measure space $(S_{\mathbf{a}}, d, \mathscr{L}^2)$ (where d denotes the Euclidean metric and $\mathscr{L}^2$ denotes the Lebesgue measure in $\mathbb{R}^2$) admits a $(1,p)$-Poincaré inequality for each $1 < p < \infty$ if $\mathbf{a} \in \ell^{\mathbf{2}}$. See [83] for these and other results.

We will consider the special case when $\mathbf{a} = (\frac{1}{a}, \frac{1}{a}, \frac{1}{a}, \ldots)$ is a constant sequence. Note that if $\mathbf{3} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\right)$, we obtain the standard Sierpiński carpet $S_{\mathbf{3}}$. Similarly, we write $S_{\mathbf{5}}$, $S_{\mathbf{7}}$, and so on, for the self-similar Sierpiński carpets defined via the constant sequences $\mathbf{5} = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \ldots\right)$, $\mathbf{7} = \left(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \ldots\right)$, and so on. See Figure V.2 for a picture of $S_{\mathbf{5}}$.

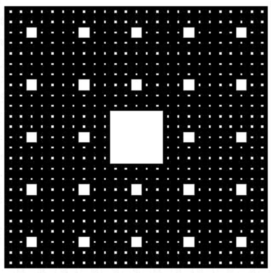


Figure V.2: Sierpiński carpet $S_{\mathbf{5}}$

(V.1.1) Coordinates in the carpet.

The easiest way to characterize points in the usual Cantor set C is via 3-adic expansions. In fact, a point x lies in C if and only if x admits a 3-adic expansion which uses no 1's.

We use the same idea to represent points in the self-similar carpet $S_{\mathbf{a}}$. Let us consider the following a-adic expansion for points $x \in \mathbb{R}$:

$$x = x_0 + \sum_{k=1}^{\infty} \frac{x_k}{a^k} \qquad x_0 \in \mathbb{Z}, x_k \in \{0, 1, \dots a-1\}. \tag{V.1}$$

In the remainder of this section, we will use the notation

$$x = (x_0.x_1|x_2|x_3|\cdots)_a \tag{V.2}$$

to denote such an expansion. In several places, we will abuse notation and express points x in the form (V.2) for positive integers x_k, $k \geq 1$, which are not necessarily in the set $\{0, 1, \dots, a-1\}$. This has the obvious interpretation as in (V.1).

We now state the desired characterization of the carpet $S_{\mathbf{a}}$.

Proposition V.1.2. *Let (x, y) be a point in $Q = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Then $(x, y) \in S_{\mathbf{a}}$ if and only if*

$$x = (0.x_1|x_2|x_3|\cdots) \qquad \textit{and} \qquad y = (0.y_1|y_2|y_3|\cdots)$$

where, for each $k \in \mathbb{N}$, either $x_k \neq (a-1)/2$ or $y_k \neq (a-1)/2$.

The proof is elementary.

V.2 Slopes of nontrivial line segments in Sierpiński carpets

Since the carpet $S_{\mathbf{a}}$ admits all of the symmetries of the unit square $\{(x, y) : 0 \leq x, y \leq 1\}$ (i.e., the dihedral group D_4), we observe that a value α occurs as a slope if and only if each of the quantities $-\alpha$, $\frac{1}{\alpha}$, and $-\frac{1}{\alpha}$ occurs as a slope (with the usual interpretation regarding 0 and ∞). Thus it suffices to characterize the slopes which lie between 0 and 1. We denote by

$$\mathrm{Slopes}(S_{\mathbf{a}})$$

the set of slopes, in the interval $[0, 1]$, of nontrivial line segments contained in the carpet $S_{\mathbf{a}}$.

The following theorem is the main result of this chapter. It characterizes self-similar carpets in terms of their slope sets, in the sense that it gives a one-to-one correspondence between self-similar carpets and the set of slopes of nontrivial line segments contained in such carpets.

Theorem V.2.1. *Let $\mathbf{a} = (\frac{1}{a}, \frac{1}{a}, \frac{1}{a}, \dots)$ be a constant sequence. Then the set of slopes* $\mathrm{Slopes}(S_{\mathbf{a}})$ *is the union of the following two sets:*

$$A = \left\{ \frac{p}{q} \,:\, p + q \leq a, \quad 0 \leq p < q \leq a - 1, \quad p, q \in \mathbb{N} \cup \{0\}, \quad p + q \textit{ odd} \right\}$$

and

$$B = \left\{ \frac{p}{q} \, : \, p + q \leq a - 1, \quad 1 \leq p \leq q \leq a - 2, \quad p, q \in \mathbb{N}, \quad p, q \text{ odd} \right\}.$$

Moreover, if $\alpha \in A$, then each nontrivial line segment in $S_{\mathbf{a}}$ with slope α touches vertices of peripheral squares, while if $\alpha \in B$, then each nontrivial line segment in $S_{\mathbf{a}}$ with slope α is disjoint from all peripheral squares. For each $\alpha \in A \cup B$, there exist maximal line segments in $S_{\mathbf{a}}$ with slope α. Finally, if $b < a$, then any maximal nontrivial line segment in $S_{\mathbf{b}}$ is also contained in $S_{\mathbf{a}}$. In particular, $\mathrm{Slopes}(S_{\mathbf{b}}) \subset \mathrm{Slopes}(S_{\mathbf{a}})$.

We say that a line segment in $S_{\mathbf{a}}$ is *maximal* if it connects two points on the boundary of the initial square $Q = \{(x, y) : 0 \leq x, y \leq 1\}$.

We list the set of slopes of the first few carpets $S_{\mathbf{a}}$. Observe that the slopes appear in strictly increasing order:

$$\mathrm{Slopes}(S_{\mathbf{3}}) = \{0, \frac{1}{2}, 1\},$$

$$\mathrm{Slopes}(S_{\mathbf{5}}) = \{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\},$$

$$\mathrm{Slopes}(S_{\mathbf{7}}) = \{0, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\},$$

and

$$\mathrm{Slopes}(S_{\mathbf{9}}) = \{0, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 1\}.$$

Remark V.2.2. If $S_{\mathbf{a}}$ contains a nontrivial line segment of some slope α ($\alpha \neq 0$), then $S_{\mathbf{a}}$ contains a nontrivial line segment of slope α which intersects the x-axis. Indeed, any line segment contained in $S_{\mathbf{a}}$ must intersect the boundary of one of the defining squares $Q_{k,j}$. Since for fixed k, all of the sets $Q_{k,j} \cap S_{\mathbf{a}}$ are isometric, there is a corresponding line segment of the same slope which intersects the boundary of the original square Q. Applying an isometry of Q if necessary, and using the invariance of the set of slopes under the operations $\alpha \mapsto -\alpha$, $\alpha \mapsto \frac{1}{\alpha}$ and $\alpha \mapsto -\frac{1}{\alpha}$, we conclude the desired fact. Figure V.3 shows nontrivial line segments of each allowed slope in the Sierpiński carpets $S_{\mathbf{3}}$ and $S_{\mathbf{5}}$.

Figure V.3 suggests the following refinement of Remark V.2.2, which is in fact correct and will be confirmed in the proof of Theorem V.2.1.

Remark V.2.3. Fix $\mathbf{a}$, write $\mathrm{Slopes}(S_{\mathbf{a}}) = A \cup B$ as in the statement of Theorem V.2.1, and fix $\alpha \in A \cup B$. If $\alpha \in A$, then there exists a line segment of slope α passing through the origin $(0, 0)$. On the other hand, if $\alpha \in B$, then there exists a line segment of slope α passing through the midpoint $(\frac{1}{2}, 0)$. Other line segments of this slope are obtained by applying Euclidean translations.

Using Remark V.2.2 we can give a quick proof that no irrational slopes can occur in any of the carpets $S_{\mathbf{a}}$.

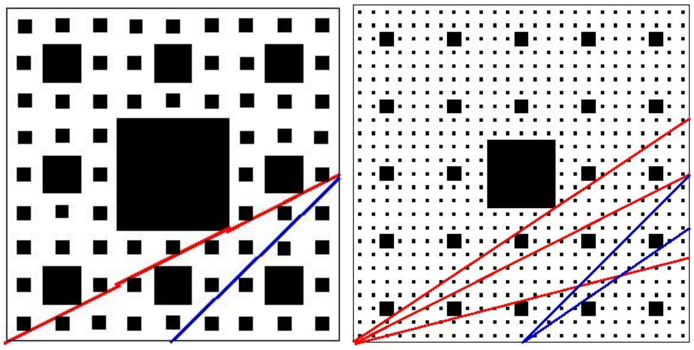


Figure V.3: Nontrivial line segments of various slopes in the carpets $S_{\mathbf{3}}$ and $S_{\mathbf{5}}$

Lemma V.2.4. *Let $S_{\mathbf{a}}$ be a carpet (possibly nonself-similar) of the type defined in section V.1. Let $\alpha \in [0,1]$ and for each point $x \in [0,1]$ consider the set*

$$A_x^\alpha = \{x + \alpha n \pmod 1 : n \in \mathbb{N}\}.$$

If each of the sets A_x^α, $x \in [0,1]$, is dense in $[0,1]$, then there is no nontrivial line segment in $S_{\mathbf{a}}$ with slope α.

Proof. By Remark V.2.2, it suffices to consider line segments meeting the x-axis.

If each of the sets A_x^α is dense in $[0,1]$, then the union of the lines with slope α through the points of A_x^α meets $[0,1]^2$ in a dense set. It follows that every nontrivial line segment through any point of the x-axis must meet complementary squares arbitrarily close to the x-axis. □

Corollary V.2.5. *There are no nontrivial line segments of irrational slope in any of the carpets $S_{\mathbf{a}}$.*

Proof. If α is irrational, then A_x^α is dense in $[0,1]$. □

Remark V.2.6. A similar argument can be used to prove that if $\mathbf{a} = (\frac{1}{a}, \frac{1}{a}, \ldots)$ for some odd integer $a \geq 3$, and if each of the sets A_x^α, $x \in [0,1]$, has no gaps of length greater than or equal than $1/a$, then there is no nontrivial line segment in $S_{\mathbf{a}}$ with slope α. However, our proof of Theorem V.2.1 will proceed along different lines.

A full proof of Theorem V.2.1 will be given in section V.2.22. In particular, we will reprove the nonexistence of line segments with irrational slope in the carpets.

(V.2.7) The set of slopes and Farey sequences.

Now, we discuss the connection between the set of slopes for a self-similar carpet $S_{\mathbf{a}}$ and Farey sequences. Our starting point is the following corollary of Theorem V.2.1.

Corollary V.2.8. *The set* $\mathrm{Slopes}(S_{\mathbf{a}})$ *contains all Farey fractions of order* $(a+1)/2$*, and is contained in the set of all Farey fractions of order* $a-1$*.*

We recall that the *Farey fractions (or Farey sequence) of order* n consist of those rational numbers in $[0,1]$ which, in lowest terms, have denominator no more than n. Farey fractions arise ubiquitously in problems at the intersection of number theory, combinatorics and geometry. Their appearance here stems from one of their well known geometric properties [92, p. 87]: *the nth Farey sequence corresponds to the integer lattice points in the triangle* $\{(x,y) : 0 \le y \le x \le n\}$ *which are directly visible from the origin.* See Remark V.2.13. For a previous use of Farey sequences in fractal geometry (enumeration of the components of the Mandelbrot set), see Devaney [33].

Proof of Corollary V.2.8. The inclusion of $\mathrm{Slopes}(S_{\mathbf{a}})$ in F_{a-1} is clear from Theorem V.2.1.

We prove the inclusion $F_{(a+1)/2} \subset \mathrm{Slopes}(S_{\mathbf{a}})$. Suppose that $\frac{p}{q}$, in lowest terms, is in $F_{(a+1)/2}$. Then $0 \le p \le q \le \frac{a+1}{2}$. If both p and q are odd, then either $p = q = 1$ or $p < q$. In the latter case, $p + q \le \frac{a+1}{2} + \frac{a-3}{2} = a - 1$. Hence $\frac{p}{q} \in B$. Suppose instead that either p or q is even. Then $0 \le p < q \le \frac{a+1}{2} \le a - 1$ (since $a \ge 3$). Furthermore, $p + q \le \frac{a+1}{2} + \frac{a-1}{2} = a$. Hence $\frac{p}{q} \in A$. □

Corollary V.2.9. $\mathrm{Slopes}(S_{\mathbf{3}}) \subsetneq \mathrm{Slopes}(S_{\mathbf{5}}) \subsetneq \mathrm{Slopes}(S_{\mathbf{7}}) \subsetneq \dots$ *and*

$$\bigcup \mathrm{Slopes}(S_{\mathbf{a}}) = [0,1] \cap \mathbb{Q}. \tag{V.3}$$

The identity in (V.3) follows from the inclusion of $F_{(a+1)/2}$ in $\mathrm{Slopes}(S_{\mathbf{a}})$. The monotonicity of the sets $\mathrm{Slopes}(S_{\mathbf{a}})$ with respect to $\mathbf{a}$ follows from the characterization in Theorem V.2.1.

As a consequence of Lemma V.2.4 and Corollary V.2.8 we draw the following interesting conclusion for Sierpiński carpets $S_{\mathbf{a}}$, when $\mathbf{a}$ is not necessarily a constant sequence.

Theorem V.2.10. *Let* $\mathbf{a} = (a_1^{-1}, a_2^{-1}, \dots) \in \left\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\right\}^{\mathbb{N}}$.

(a) *If* $\limsup \mathbf{a} = 0$ *(i.e., if* $\mathbf{a} \in c_0$*), then* $S_{\mathbf{a}}$ *contains nontrivial line segments of every rational slope, and contains no nontrivial line segments of any irrational slope.*

(b) *If* $\limsup \mathbf{a} > 0$*, then* $\limsup \mathbf{a} = \frac{1}{a_0}$ *for some* $a_0 \in \{3, 5, 7, \cdots\}$*. In this case,* $\mathrm{Slopes}(S_{\mathbf{a}})$ *coincides with* $\mathrm{Slopes}(S_{\mathbf{a_0}})$*.*

Proof. If $\limsup \mathbf{a} = 0$, then $\lim \mathbf{a} = 0$ and $\lim_{k\to\infty} a_k = \infty$. By Corollary V.2.9 and Theorem V.2.1, if $a_k \ge b$ for all sufficiently large k, then all corresponding subsquares $Q_{k,j} \cap S_{\mathbf{a}}$ contain nontrivial line segments of all slopes α in $\mathrm{Slopes}(S_{\mathbf{b}})$. Since b may be chosen arbitrarily large and every positive rational is a Farey fraction of some order, the statement in part (a) follows. The second statement follows from Lemma V.2.4.

For the proof of part (b), we note that if $\limsup \mathbf{a} > 0$ and $a_0 = \min\{\frac{1}{b} : b \in \mathbf{a}\}$, then $a_k = a_0$ for infinitely many values of k. From the fact that $a_k = a_0$, we easily deduce that there are no line segments with slope not in $\mathrm{Slopes}(S_{\mathbf{a_0}})$ whose length exceeds some quantity ϵ_k, where $\epsilon_k \to 0$ as $k \to \infty$. Hence there are no nontrivial line segments in $S_{\mathbf{a}}$ with slopes which are not in $\mathrm{Slopes}(S_{\mathbf{a_0}})$. We postpone discussion of the remaining claim (there exist nontrivial line segments in $S_{\mathbf{a}}$ with each slope in $\mathrm{Slopes}(S_{\mathbf{a_0}})$) to Remark V.2.24. □

Remark V.2.11. Lemma V.2.4 can also be explained by the aid of the theory of square billiards [27]. Consider a square billiard table Q and a particle moving inside Q. When the moving particle reaches the boundary ∂Q, the angle of incidence is equal to the angle of reflection. However, instead of reflecting the trajectory of the particle in a side of ∂Q, let us reflect the square Q across that side and allow the particle to move straight into the mirror image of Q. If we repeat this procedure at every collision, the particle will move along a straight line through multiple copies of Q obtained by successive reflections. This construction is called *unfolding* the billiard trajectory. To recover the original trajectory in Q, one *folds* the resulting string of adjacent copies of Q back onto Q. If we consider the 2×2 square

$$Q_2 = \{(x, y) : 0 \le x \le 2, 0 \le y \le 2\},$$

the standard projection of $\mathbb{R}^2$ onto Q_2 transforms unfolded trajectories into directed straight lines on the 2×2 torus (the latter is obtained by identifying opposite sides of the square Q_2). Billiards in the square thus reduces to simple linear flow on a torus. The linear flow on a flat torus is one of the standard examples in ergodic theory. Its main properties are:

- if a trajectory has rational slope, then it is periodic (it runs along a closed geodesic),
- if a trajectory has irrational slope, then it is dense (its closure is the whole torus).

The theory of square billiards can be applied to study line segments contained in the carpets $S_{\mathbf{a}}$. Instead of considering a square, we consider a "punctured" square, and so a punctured torus. Here by "punctured" we mean a closed square with a square hole in the center of the corresponding size $\frac{1}{a}$. According to the above results, trajectories with irrational slope can not occur in the punctured torus either. However, since we now have a hole, not all rational slopes will occur, since eventually the trajectory will hit the hole. In this way, Theorem V.2.1 can be interpreted as a game of "punctured" squared billiards.

The relationship between line segments in the carpet and the dynamics of the corresponding square billiards is made somewhat more precise in Proposition V.2.19, which gives a criterion for membership in $\mathrm{Slopes}(S_{\mathbf{a}})$.

Remark V.2.12. Boca, Gologan and Zaharescu [14], [15] already used the Farey sequences to study the statistics of the first exit time and collision number for punctured toral billiards with circular punctures (which in turn can be used to model the periodic 2D Lorentz gas).

Remark V.2.13. We indicate a more geometric way to look at the set of slopes which illuminates the connection to Farey sequences. First, let us introduce a bijection between

$$Z = \{(q,p) \in \mathbb{N}^2 : p \text{ and } q \text{ are coprime}\}$$

and the positive rationals by the rule $\varphi : (q,p) \mapsto \frac{p}{q}$. Consider the set Z' consisting of all elements (q,p) of Z satisfying $p+q \leq a$ and $p \leq q$. Then $\mathrm{Slopes}(S_{\mathbf{a}}) = \{0\} \cup \varphi(Z')$. This follows directly from Theorem V.2.1. See Figure V.4.

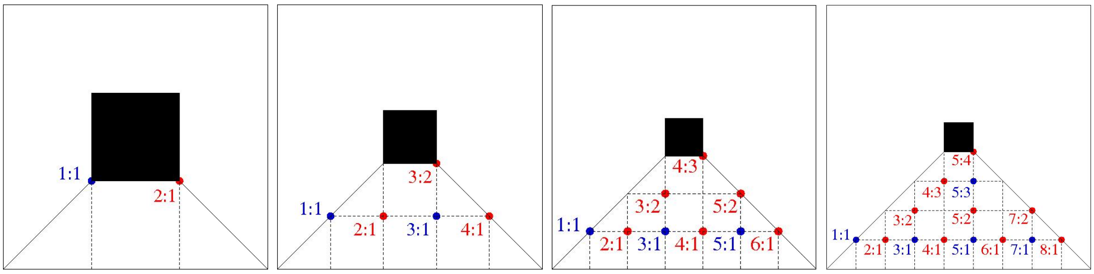


Figure V.4: Pictorial representation of the slope set for the carpets $S_{\mathbf{a}}$, $\mathbf{a} \in \{\mathbf{3}, \mathbf{5}, \mathbf{7}, \mathbf{9}\}$

Remark V.2.14. The inclusion of $\mathrm{Slopes}(S_{\mathbf{a}})$ in F_{a-1} will not be directly useful for us since $\mathrm{Slopes}(S_{\mathbf{a}})$ does not appear in F_{a-1} as a consecutive block of elements, that is, there exist elements in $F_{a-1} \setminus \mathrm{Slopes}(S_{\mathbf{a}})$ which lie between two elements of $\mathrm{Slopes}(S_{\mathbf{a}})$. In order to take advantage of properties of Farey sequences we will give another description of $\mathrm{Slopes}(S_{\mathbf{a}})$.

For each odd a, consider the finite set of fractions

$$f_n = \frac{n}{a-1-n}, \qquad n = 0, \ldots, \frac{a-1}{2}.$$

Observe that $\{f_n\}_n \in \mathrm{Slopes}(S_{\mathbf{a}})$. Under the bijection φ from Remark V.2.13, this set corresponds to lattice points which appear just below the "main diagonal", that is, points in the segment which connects $(a-1, 0)$ to $(\frac{a-1}{2}, \frac{a-1}{2})$.

Next, consider the following inclusion between ordered sets:

$$\psi : (\{f_n\}_n, \leq) \longrightarrow (\mathrm{Slopes}(S_{\mathbf{a}}), \leq).$$

Let $\mathrm{Slopes}(S_{\mathbf{a}}) = \{0 = s_0, \cdots, s_r = 1\}$ (in increasing order) and define

$$\phi : \{0, \ldots, \frac{a-1}{2}\} \longrightarrow \{0, \cdots, r\}$$

by setting $\phi(n) = j$ if and only if $\psi(f_n) = s_j$. The set of slopes $\mathrm{Slopes}(S_{\mathbf{a}})$ can be written as the union of

$$[s_0 = s_{\phi(0)}, s_{\phi(1)}] \cap \mathrm{Slopes}(S_{\mathbf{a}}),$$

$$[s_{\phi(1)}, s_{\phi(2)}] \cap \mathrm{Slopes}(S_{\mathbf{a}}),$$

and so on, through

$$[s_{\phi(\frac{a-3}{2})}, s_{\phi(\frac{a-1}{2})} = s_r] \cap \mathrm{Slopes}(S_{\mathbf{a}}).$$

Proposition V.2.15. *For each $n = 1, \ldots, \frac{a-1}{2}$, the set*

$$[s_{\phi(n-1)}, s_{\phi(n)}] \cap \mathrm{Slopes}(S_{\mathbf{a}})$$

is a sequence of consecutive elements in F_{a-n}.

Proof. Let $n = 1, \ldots, \frac{a-1}{2}$. It suffices to prove that

$$[s_{\phi(n-1)}, s_{\phi(n)}] \cap \mathrm{Slopes}(S_{\mathbf{a}}) = [s_{\phi(n-1)}, s_{\phi(n)}] \cap F_{a-n}. \tag{V.4}$$

Let $\frac{p}{q}$ be a rational number expressed in lowest terms and satisfying

$$\frac{n-1}{a-n} \leq \frac{p}{q} \leq \frac{n}{a-1-n}. \tag{V.5}$$

The identity in (V.4) asserts that under these hypotheses,

$$0 \leq p \leq q \leq a-1 \quad \text{and} \quad p+q \leq a \qquad \text{if and only if} \qquad 0 \leq p \leq q \leq a-n.$$

First, we prove the "only if" statement. Assume that $0 \leq p \leq q \leq a-1$ and $p+q \leq a$. The conclusion being obvious if $n = 1$, we also assume that $n \geq 2$. Then from (V.5) we obtain

$$(a-1)q = (a-n+n-1)q \leq (a-n)(p+q) \leq a(a-n)$$

so

$$q \leq \frac{a(a-n)}{a-1} < a-n+1$$

(since $n \geq 2$). Since q is an integer, we must have $q \leq a-n$.

Next, we prove the "if" statement. Assume that $0 \leq p \leq q \leq a-n$. Then from (V.5) we obtain

$$(a-1-n)(p+q) \leq (a-1)q \leq (a-n)(a-1)$$

so

$$p + q \leq \frac{(a-n)(a-1)}{a-1-n} \leq a + 1. \tag{V.6}$$

If strict inequality holds in either place in (V.6), then $p + q \leq a$, since $p + q$ is an integer. Otherwise, $n = \frac{a-1}{2}$ and $a + 1 = p + q \leq 2q \leq 2(a - n) = a + 1$ which yields $p = q = \frac{a+1}{2}$. This contradicts the initial assumption that $\frac{p}{q}$ is in lowest terms (recall that $a \geq 3$). $\square$

Proposition V.2.15 asserts that $\text{Slopes}(S_{\mathbf{a}})$ can be written as a union of "intervals", each of which consists of a consecutive block within a particular Farey sequence. This observation allows us to use many of the properties of the Farey sequences in the proof of Theorem V.2.1. We enumerate here some of the more remarkable properties of Farey sequences. See [27, Ch. 3].

Proposition V.2.16. *Farey sequences enjoy the following properties:*

(1) *$F_n \subset F_{n+1}$. If $p_1/q_1 < p_2/q_2$ are consecutive in F_n and separated in F_{n+1}, then the fraction $\frac{p_1+p_2}{q_1+q_2}$ lies in between p_1/q_1 and p_2/q_2 and no other elements of F_{n+1} lies between p_1/q_1 and p_2/q_2. The fraction $\frac{p_1+p_2}{q_1+q_2}$ is called the* mediant *of p_1/q_1 and p_2/q_2.*

(2) *If p_1/q_1 and p_2/q_2 are consecutive in any F_n, then they satisfy the unimodular relation $p_1 \cdot q_2 = p_2 \cdot q_1 - 1$.*

Observe that the mediant of consecutive Farey fractions is already in reduced form. Indeed, suppose that p/q is the mediant of p_1/q_1 and p_2/q_2, and that p_1/q_1 and p_2/q_2 are consecutive Farey fractions of some order. Then $p_2q - q_2p \geq 1$ and $pq_1 - qp_1 \geq 1$. Furthermore, by Proposition V.2.16(2), we have

$$q_1 + q_2 = q = q_1(p_2q - q_2p) + q_2(pq_1 - qp_1) \geq q_1 + q_2$$

which shows that $p_2q - q_2p = pq_1 - qp_1 = 1$. By Euclid's algorithm, p and q are coprime.

The following lemma provides us a recursive way to construct the set of slopes.

Lemma V.2.17. *Suppose p_1/q_1 and p_2/q_2 are consecutive fractions in* $\text{Slopes}(S_{\mathbf{a}})$*, both in reduced form. Then p_1/q_1 and p_2/q_2 are separated in* $\text{Slopes}(S_{\mathbf{a+2}})$ *if and only if $p_1 + q_1 + p_2 + q_2 \leq a + 2$.*

Proof. Recall (see Remark V.2.13) that

$$\text{Slopes}(S_{\mathbf{a}}) = \{(q, p) \in \mathbb{Z}^2 : p + q \leq a,\ 0 \leq p \leq q\}. \tag{V.7}$$

Since p_1/q_1 and p_2/q_2 are consecutive in $\text{Slopes}(S_{\mathbf{a}})$, they are consecutive in some F_n and hence their mediant is in reduced form.

First let us assume that p_1/q_1 and p_2/q_2 are separated in Slopes($S_{\mathbf{a+2}}$). Then the mediant

$$\frac{p}{q} = \frac{p_1 + p_2}{q_1 + q_2}$$

appears in Slopes($S_{\mathbf{a+2}}$), and so $p + q = p_1 + q_1 + p_2 + q_2 \leq a + 2$. On the other hand, if p_1/q_1 and p_2/q_2 are consecutive fractions in Slopes($S_{\mathbf{a+2}}$), then the fraction $\frac{p}{q} = \frac{p_1+p_2}{q_1+q_2}$ is not in Slopes($S_{\mathbf{a+2}}$) and so, by (V.7), we conclude that $p_1+q_1+p_2+q_2 = p+q > a+2$. $\square$

Observe that one can generate Slopes($S_{\mathbf{a+2}}$) from Slopes($S_{\mathbf{a}}$) just by adding the medians of those consecutive fractions p_1/q_1 and p_2/q_2 in Slopes($S_{\mathbf{a}}$) for which $p_1+q_1+p_2+q_2 \leq a + 2$. Notice also that between $0/1$ and $1/(a-1)$ there always appear two fractions in Slopes($S_{\mathbf{a+2}}$). The reason is simple. In this case, the mediant of $0/1$ and $1/(a-1)$ is $1/a$. However, there is still space between $p_1/q_1 = 0/1$ and $p_2/q_2 = 1/a$, since $p_1 + q_1 + p_2 + q_2 = a + 1 < a + 2$. Thus, the fractions $1/a$ and $1/(a+1)$ appear between $0/1$ and $1/(a-1)$.

(V.2.18) A necessary condition for a line segment to lie in the carpet $S_{\mathbf{a}}$.

To show that the values in Slopes($S_{\mathbf{a}}$) are the only slopes which occur, we will need the following useful criterion. The idea of this criterion is that *going deeper into the carpet* corresponds to *tiling the plane with squares.* This is closely related to the interpretation of line segments in the carpet in terms of square billiards, as in Remark V.2.11.

Proposition V.2.19. *If there exists a nontrivial line segment of a certain slope α emanating from a point $(c, 0)$, $c \in [0, 1]$, and contained in the carpet $S_{\mathbf{a}}$, that is, if the set*

$$L_{c,\alpha}^{S_{\mathbf{a}}} = \{(x, y) \in S_{\mathbf{a}} : y = \alpha(x - c)\}$$

contains a line segment containing $(c, 0)$, then the line

$$L_{c,\alpha} = \{(x, y) \in \mathbb{R}^2 : y = \alpha(x - c)\}$$

does not intersect any member of the planar tiling

$$\mathbb{Z}^2 + Q_{\mathbf{a}} := \left\{(k, \ell) + Q_{\mathbf{a}} : (k, \ell) \in \mathbb{Z}^2\right\},$$

where $Q_{\mathbf{a}} = \{(x, y) \in \mathbb{R}^2 : \frac{a-1}{2a} < x < \frac{a+1}{2a},\ \frac{a-1}{2a} < y < \frac{a+1}{2a}\}$.

We sketch the proof of Proposition V.2.19. Suppose that we are at the mth level of the construction of $S_{\mathbf{a}}$. Replace each square that we have removed in all the previous steps by a concentric square of side length a^{-m} and called the resulting set A_m. Observe that $A_m \supset S_{\mathbf{a}}$. If $L_{c,\alpha}^{S_{\mathbf{a}}}$ contains a line segment containing $(c, 0)$, then that line segment also lies in the sets A_m for each $m \in \mathbb{N}$. The conclusion now follows by rescaling and passing to the limit as m tends to infinity.

Corollary V.2.20. *If there exists a nontrivial line segment of a certain slope α emanating from a point $(c,0)$, $c \in [0,1]$, and contained in the carpet $S_{\mathbf{a}}$, then the line $L_{0,\alpha}$ emanating from the origin of slope α does not intersect the planar tiling*

$$a\mathbb{Z}^2 + Q' := \left\{(ak, a\ell) + Q' : (k,\ell) \in \mathbb{Z}^2\right\}, \tag{V.8}$$

where $Q' = \{(x,y) : -1 < x < 0,\, 0 < y < 1\}$.

We indicate how Corollary V.2.20 follows from Proposition V.2.19. First, apply the homothety $(x,y) \mapsto (ax - \frac{a+1}{2}, ay - \frac{a-1}{2})$. Then some line $L_{c',\alpha}$ of slope α does not intersect the tiling $a\mathbb{Z}^2 + Q'$. If $L_{c',\alpha}$ passes through the inferior right vertex of any of the squares in $a\mathbb{Z}^2 + Q'$, then applying another translation shows that the line through the origin of slope α also does not intersect the tiling. If $L_{c',\alpha}$ does not pass through the inferior right vertex of any square in $a\mathbb{Z}^2 + Q'$, we distinguish two cases:

(i) The distance from $L_{c',\alpha}$ to the set S of all inferior right vertices of squares in $a\mathbb{Z}^2 + Q'$ is positive. In this case, identify a vertex v in S whose distance to $L_{c',\alpha}$ is minimal. Translate $L_{c',\alpha}$ to pass through v; such translation does not affect the fact that this line does not intersect the tiling. Finally, translating v to the origin completes the proof.

(ii) The distance from $L_{c',\alpha}$ to S is equal to zero, but is not achieved. Choose a sequence of vertices (v_n) in S such that $\operatorname{dist}(L_{c',\alpha}, v_n) \to 0$. Applying the corresponding sequence of translations (which take these points successively to the origin) yields a sequence of lines, all of slope α, which do not intersect the tiling and whose distance to the origin tends to zero. The limiting line also has slope α, passes through the origin, and does not intersect the tiling.

Remark V.2.21. We emphasize a subtle point in the preceding argument. Consider the decomposition $\mathrm{Slopes}(S_{\mathbf{a}}) = A \cup B$ associated to a specific self-similar carpet $S_{\mathbf{a}}$. For α in B, as already mentioned, there are no lines of slope α which meet any of the vertices of the peripheral squares associated to $S_{\mathbf{a}}$. However, there do exist such lines passing through vertices of squares associated to the corresponding tiling of the plane given in Corollary V.2.20. The reason is that this tiling is not a self-similar fractal construction but rather has a definite lower scale; all of the squares in the tiling have mutual distance at least one.

(V.2.22) Proof of the main theorem.

We are now in a position to prove Theorem V.2.1. We divide the proof into two parts. In the first part, we show that nontrivial line segments exist whenever the slope α is chosen from the set $A \cup B$. In the second part, we show that no other slopes occur.

Part 1. Let $\alpha \in A \cup B$. The strategy of this part of the proof is to use the carpet coordinates introduced in section V.1.1 to see that the lines $y = \alpha(x - c)$ do not intersect the omitted open squares. It is important to note here that if the line segment L has slope $\alpha \in A$, then we can assume that $c = 0$, that is, L emanates from a vertex. On the other hand, if L has slope $\alpha \in B$, then we can assume $c = \frac{1}{2}$, that is, L emanates from the midpoint.

Observe that if $(x, y) \notin Q_n$ for some $n \in \mathbb{N}$, i.e., if (x, y) is contained in some omitted square, then

$$\left(0.x_1\middle|\cdots\middle|x_{n-1}\middle|\frac{a-1}{2}\middle|0\middle|0\middle|\cdots\right)_a < x < \left(0.x_1\middle|\cdots\middle|x_{n-1}\middle|\frac{a+1}{2}\middle|0\middle|0\middle|\cdots\right)_a \tag{V.9}$$

and

$$\left(0.y_1|\cdots\middle|y_{n-1}\middle|\frac{a-1}{2}\middle|0\middle|0\middle|\cdots\right)_a < y < \left(0.y_1|\cdots\middle|y_{n-1}\middle|\frac{a+1}{2}\middle|0\middle|0\middle|\cdots\right)_a. \tag{V.10}$$

The proof will involve detailed computations and estimates of the coordinates of points in base a, comparing the condition for membership in one of the omitted squares with membership in the line L.

Case 1a: $\alpha \in A$. We claim that the line L given by the equation

$$y = \alpha x$$

does not meet any of the omitted squares from the construction of $S_{\mathbf{a}}$.

Suppose that (x, y) is a point contained in some omitted square and also contained in L. Since $\alpha \in A$, there exist $p, q \in \mathbb{N} \cup \{0\}$ with $p + q$ odd, $p + q \leq a$, $0 \leq p < q \leq a - 1$, and

$$qy = px.$$

If we multiply by p in (V.9) and by q in (V.10), we obtain

$$\left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\widetilde{x_{n-1}}\middle|\frac{(a-1)p}{2}\middle|0\middle|\cdots\right)_a < px < \left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\widetilde{x_{n-1}}\middle|\frac{(a+1)p}{2}\middle|0\middle|\cdots\right)_a \tag{V.11}$$

and

$$\left(\widetilde{y_0}.\widetilde{y_1}\middle|\cdots\middle|\widetilde{y_{n-1}}\middle|\frac{(a-1)q}{2}\middle|0\middle|\cdots\right)_a < qy < \left(\widetilde{y_0}.\widetilde{y_1}\middle|\cdots\middle|\widetilde{y_{n-1}}\middle|\frac{(a+1)q}{2}\middle|0\middle|\cdots\right)_a \tag{V.12}$$

respectively. Observe that coordinates are written modulo a, and we employ the previously mentioned abuse of notation (the coefficients need not be integers in the range $\{0, 1, \ldots, a-1\}$). Moreover, it follows from (V.11) that $p \neq 0$.

If p is even we make a simple arithmetic calculation to recast (V.11) and (V.12) as follows:

$$\left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\frac{p}{2}-1+\widetilde{x_{n-1}}\middle|a-\frac{p}{2}\middle|0\middle|\cdots\right)_a < px < \left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\frac{p}{2}+\widetilde{x_{n-1}}\middle|\frac{p}{2}\middle|0\middle|\cdots\right)_a$$

and

$$\left(\widetilde{y_0}.\widetilde{y_1}\middle|\cdots\middle|\frac{q-1}{2}+\widetilde{y_{n-1}}\middle|\frac{a-q}{2}\middle|0\middle|\cdots\right)_a < qy < \left(\widetilde{y_0}.\widetilde{y_1}\middle|\cdots\middle|\frac{q-1}{2}+\widetilde{y_{n-1}}\middle|\frac{a+q}{2}\middle|0\middle|\cdots\right)_a.$$

Since $p \geq 2$ and $q \leq a-2$ (note that q is odd), we observe that we have reduced the nth coefficients to the range $\{0,1,\ldots,a-1\}$. We next observe that $\frac{p}{2} \leq \frac{a-q}{2}$ and $\frac{a+q}{2} \leq a-\frac{p}{2}$ by the conditions on p and q. Since the $(n-1)$st coefficients in the bounds for qy are equal, while the $(n-1)$st coefficients in the bounds for px disagree by one, we conclude that no such point (x,y) can exist. Similarly, if p is odd, we recast (V.11) and (V.12) as follows:

$$\left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\frac{p-1}{2}+\widetilde{x_{n-1}}\middle|\frac{a-p}{2}\middle|0\middle|\cdots\right)_a < px < \left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\frac{p-1}{2}+\widetilde{x_{n-1}}\middle|\frac{a+p}{2}\middle|0\middle|\cdots\right)_a$$

and

$$\left(\widetilde{y_0}.\widetilde{y_1}\middle|\cdots\middle|\frac{q-2}{2}+\widetilde{y_{n-1}}\middle|a-\frac{q}{2}\middle|0\middle|\cdots\right)_a < qy < \left(\widetilde{y_0}.\widetilde{y_1}\middle|\cdots\middle|\frac{q}{2}+\widetilde{y_{n-1}}\middle|\frac{q}{2}\middle|0\middle|\cdots\right)_a.$$

Since $q \geq 2$ and $p \leq a-2$ (note that p is odd), we observe that we have reduced the nth coefficients to the range $\{0,1,\ldots,a-1\}$. We next observe that $\frac{a+p}{2} \leq a-\frac{q}{2}$ and $\frac{q}{2} \leq \frac{a-p}{2}$ by the conditions on p and q. Since the $(n-1)$st coefficients in the bounds for px are equal, while the $(n-1)$st coefficients in the bounds for qy disagree by one, we conclude that no such point (x,y) can exist.

Case 1b: $\alpha \in B$. We claim that the line L given by the equation

$$y = \alpha(x-\frac{1}{2})$$

does not meet any of the omitted squares from the construction of $S_{\mathbf{a}}$.

Suppose that (x,y) is a point contained in some omitted square and also contained in L. Since $\alpha \in B$, there exist odd integers $p,q \in \mathbb{N}$ with $p+q \leq a-1$, $1 \leq p \leq q \leq a-2$, and

$$\frac{p}{2}+qy = px.$$

Note that

$$\frac{p}{2} = \frac{p-1}{2}+\frac{1}{2} = \left(\frac{p-1}{2}.\frac{a-1}{2}\middle|\frac{a-1}{2}\middle|\frac{a-1}{2}\middle|\cdots\right)_a. \tag{V.13}$$

If we multiply by p in (V.9), by q in (V.10) and add $\frac{p}{2}$ (written in the form (V.13)) to the latter, we obtain

$$\left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\widetilde{x_{n-1}}\middle|\frac{(a-1)p}{2}\middle|0\middle|\cdots\right)_a < px < \left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\widetilde{x_{n-1}}\middle|\frac{(a+1)p}{2}\middle|0\middle|\cdots\right)_a \qquad \text{(V.14)}$$

and

$$\begin{gathered}\left(\frac{p-1}{2}+\widetilde{y_0}.\frac{a-1}{2}+\widetilde{y_1}\middle|\cdots\middle|\frac{a-1}{2}+\widetilde{y_{n-1}}\middle|\frac{(a-1)(q+1)}{2}\middle|\frac{a-1}{2}\middle|\cdots\right)_a\\ <\frac{p}{2}+qy< \qquad \text{(V.15)}\\ \left(\frac{p-1}{2}+\widetilde{y_0}.\frac{a-1}{2}+\widetilde{y_1}\middle|\cdots\middle|\frac{a-1}{2}+\widetilde{y_{n-1}}\middle|\frac{(a-1)+(a+1)q}{2}\middle|\frac{a-1}{2}\middle|\cdots\right)_a\end{gathered}$$

respectively. Note that (V.14) coincides with (V.11), while (V.15) is the sum of (V.12) and (V.13).

Another simple arithmetic calculation recasts (V.14) and (V.15) as follows:

$$\left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\frac{p-1}{2}+\widetilde{x_{n-1}}\middle|\frac{a-p}{2}\middle|0\middle|\cdots\right)_a < px < \left(\widetilde{x_0}.\widetilde{x_1}\middle|\cdots\middle|\frac{p-1}{2}+\widetilde{x_{n-1}}\middle|\frac{a+p}{2}\middle|0\middle|\cdots\right)_a$$

and

$$\begin{gathered}\left(\frac{p-1}{2}+\widetilde{y_0}.\frac{a-1}{2}+\widetilde{y_1}\middle|\cdots\middle|\frac{a+q-2}{2}+\widetilde{y_{n-1}}\middle|a-\frac{q+1}{2}\middle|\frac{a-1}{2}\middle|\cdots\right)_a\\ <\frac{p}{2}+qy<\\ \left(\frac{p-1}{2}+\widetilde{y_0}.\frac{a-1}{2}+\widetilde{y_1}\middle|\cdots\middle|\frac{a+q}{2}+\widetilde{y_{n-1}}\middle|\frac{q-1}{2}\middle|\frac{a-1}{2}\middle|\cdots\right)_a.\end{gathered}$$

Note that we have reduced the nth coefficients to the range $\{0, 1, \ldots, a-1\}$. We now observe that $\frac{a+p}{2} \leq a - \frac{q+1}{2}$ and $\frac{q-1}{2} \leq \frac{a-p}{2}$ by the conditions on p and q. Since the $(n-1)$st coefficients in the bounds for px are equal, while the $(n-1)$st coefficients in the bounds for $\frac{p}{2} + qy$ disagree by one, we conclude that no such point (x, y) can exist.

Note that in Case 1b there is a definite gap between the ranges of possible values for px and $\frac{p}{2} + qy$. This gap corresponds to the fact that lines with slope in B avoid all of the peripheral squares in the construction of the carpet.

This completes the proof of Part 1.

Remark V.2.23. An analysis of the preceding proof confirms the previous assertion that every maximal line segment contained in a carpet $S_{\mathbf{a}}$ is also contained in carpets $S_{\mathbf{b}}$ for $b \geq a$. Suppose that α is a slope in either of the sets A or B, associated to $\mathrm{Slopes}(S_{\mathbf{a}})$. If $b \geq a$, we may repeat the arithmetic calculations of the preceding proofs, working modulo b

instead of modulo a. The conclusions remain the same. We conclude that the appropriate line segments $\{(x,y) \in Q : y = \alpha x\}$ or $\{(x,y) \in Q : y = \alpha(x - \frac{1}{2})\}$ persist as subsets of $S_{\mathbf{b}}$.

Remark V.2.24. A straightforward variation on the above proof shows that $S_{\mathbf{a}}$ contains nontrivial line segments of each slope in $\mathrm{Slopes}(S_{a_0})$ whenever $\limsup \mathbf{a} = \frac{1}{a_0} > 0$.

Part 2. Now let $\alpha \notin A \cup B$. We claim that there is no nontrivial line segment of slope α contained in $S_{\mathbf{a}}$. By Corollary V.2.20, it suffices to show that the line L_α of slope α passing through the origin intersects the planar tiling $a\mathbb{Z}^2 + Q'$ given in (V.8).

Observe that lines through the origin which pass through the inferior right vertex of any square in the tiling have slope $a\ell/ak = \ell/k$ for some $k, \ell \in \mathbb{Z}$. On the other hand, the slope of any line through the origin which passes through the superior left vertex of any such square has slope $(a\ell + 1)/(ak - 1)$ for some $k, \ell \in \mathbb{Z}$. Consequently, the line L_β of slope β passing through the origin intersects a square from the tiling if and only if

$$\frac{\ell}{k} < \beta < \frac{a\ell + 1}{ak - 1}$$

for some relatively prime integers $0 \le \ell < k$.

We may choose consecutive slopes p_1/q_1 and p_2/q_2 in $\mathrm{Slopes}(S_{\mathbf{a}})$ so that

$$\frac{p_1}{q_1} < \alpha < \frac{p_2}{q_2}. \tag{V.16}$$

Now, for each $n \ge 0$, define the *iterated mediants*

$$\alpha_n = \frac{p_1 + np_2}{q_1 + nq_2}.$$

Note that α_{n+1} is the mediant of α_n and p_2/q_2. All of these rational numbers are in reduced form. We claim that the union of the intervals

$$\alpha_n = \frac{p_1 + np_2}{q_1 + nq_2} < \beta < \frac{a(p_1 + np_2) + 1}{a(q_1 + nq_2) - 1}, \qquad n \ge 0, \tag{V.17}$$

covers the interval (V.16). Thus α is contained in one of the intervals (V.17), and hence the line L_α must intersect one of the squares from the tiling. See Figure V.5.

Since the iterated mediants α_n converge to p_2/q_2 as $n \to \infty$, it suffices to prove that

$$\frac{p_1 + (n-1)p_2}{q_1 + (n-1)q_2} < \frac{p_1 + np_2}{q_1 + nq_2} < \frac{a(p_1 + (n-1)p_2) + 1}{a(q_1 + (n-1)q_2) - 1}$$

for each n. The left hand inequality follows immediately from (V.16). After some computations, the right hand inequality is equivalent to

$$a(q_1 p_2 - p_1 q_2) < n(p_2 + q_2) + p_1 + q_1,$$

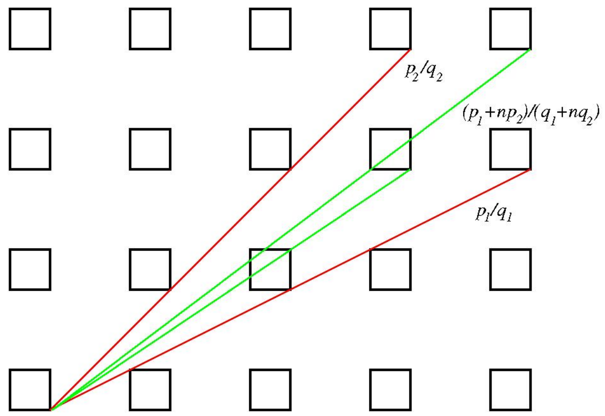


Figure V.5: Lines with iterated mediant slopes

for each $n \in \mathbb{N}$. By property (2) in Proposition V.2.16 we know that $q_1p_2 - p_1q_2 = 1$, so we only have to prove that

$$a < p_2 + q_2 + p_1 + q_1. \tag{V.18}$$

Since p_1/q_1 and p_2/q_2 are consecutive fractions in Slopes($S_{\mathbf{a}}$), inequality (V.18) can be deduced from Lemma V.2.17 and the result follows.

V.3 Differentiable and rectifiable curves in the carpets

Characterizing the slopes of line segments which occur in the carpet permits us to draw conclusions regarding the set of differentiable curves in the carpet. For instance, since the set of slopes has no interior, we easily see that there are no C^1 curves contained in any of the carpets $S_{\mathbf{a}}$ except for the line segments. We now extend this statement to cover all differentiable curves.

Proposition V.3.1. *Let* $\mathbf{a}$ *be any sequence in* $\{\frac{1}{3}, \frac{1}{5}, \ldots\}^{\mathbb{N}}$. *Every curve* $\gamma \subset S_{\mathbf{a}}$ *which is differentiable with nonzero derivative everywhere is a line segment.*

The partial derivatives of such a curve satisfy the Darboux property.

Definition V.3.2. A real-valued function f defined on an interval I satisfies the *Darboux property* if f takes every connected set to a connected set.

Let $\gamma = (x, y)$ be a curve as in Proposition V.3.1. Without loss of generality, we may assume that the curve is parametrized to have speed one everywhere: $x'(t)^2 + y'(t)^2 \equiv 1$.

A simple argument using the Darboux property shows that the range of $\gamma' = (x', y')$ is a connected subset of $\mathbb{S}^1$. Since the slope of the tangent vector at time t is given by

$$\alpha(t) = \frac{y'(t)}{x'(t)},$$

we conclude that the range of α is connected. Since the set of slopes has no interior, we conclude the proof of Proposition V.3.1 modulo the following lemma.

Lemma V.3.3. *Let γ be a differentiable curve in $S_{\mathbf{a}}$. Then $\gamma'(t) \in \mathrm{Slopes}(S_{\mathbf{a}})$ for all t.*

Lemma V.3.3 is proved by Bandt and Mubarak in [8] in the case $\mathbf{a} = (\frac{1}{3}, \frac{1}{3}, \dots)$ and the general case is similar. Here we provide only a sketch. The proof uses the following quantitative version of the fact that $S_{\mathbf{a}}$ contains no nontrivial segments with slopes which are not in $\mathrm{Slopes}(S_{\mathbf{a}})$:

> *If $\alpha \notin \mathrm{Slopes}(S_{\mathbf{a}})$ and L denotes a nontrivial line segment of slope α through a point $(x, y) \in S_{\mathbf{a}}$, then for all sufficiently small ϵ there exists a point (x', y') in $B((x, y), \epsilon) \cap L$ whose distance to $S_{\mathbf{a}}$ is at least $c\epsilon$, where $c > 0$ depends only on $\mathrm{dist}(\alpha, \mathrm{Slopes}(S_{\mathbf{a}}))$.*

Suppose that there exists a differentiable curve γ contained entirely in $S_{\mathbf{a}}$, and $\gamma'(t) \notin \mathrm{Slopes}(S_{\mathbf{a}})$ for some time t. Then $\gamma(s)$ is well approximated by $\gamma(t) + (s - t)\gamma'(t)$ for s near t and hence the line segment $s \mapsto \gamma(t) + (s - t)\gamma'(t)$ remains close to the carpet $S_{\mathbf{a}}$ for s near t. This can be used eventually to contradict the preceding quantitative statement.

To conclude the chapter, we would like to point out that the typical point of $S_{\mathbf{3}}$ lies in no nontrivial line segment contained in $S_{\mathbf{3}}$. Indeed, let $\alpha_C = \log(2)/\log(3)$ and $\alpha_S = \log(8)/\log(3)$. It is well known that α_C is the Hausdorff dimension of the Cantor set C, and α_S the Hausdorff dimension of the Sierpinski carpet $S_{\mathbf{3}}$. Observe that the set $[0, 1] \times C \subset S_{\mathbf{3}}$ has Hausdorff dimension $\alpha_C + 1$ (see for example [41, 4.3]). Moreover, the union of all nontrivial line segments contained in $S_{\mathbf{3}}$ has Hausdorff dimension $\alpha_C + 1$, which is strictly less than α_S, the Hausdorff dimension of $S_{\mathbf{3}}$.

Chapter VI

Appendix: Metric differentiability of Lipschitz maps on Wiener spaces

This last chapter is devoted to the differentiability properties of $\mathcal{H}$-Lipschitz maps defined on abstract Wiener spaces and with values in metric spaces. The classical Rademacher theorem states that any Lipschitzian mapping f from $\mathbb{R}^n$ to $\mathbb{R}^k$ is Frechét differentiable almost everywhere, with respect to the Lebesgue measure. However, this result has no direct extensions to the infinite dimensional case for two main reasons. The first one, is the lack of infinite dimensional analogues of Lebesgue measure. The second one, is the existence of Lipschitzian mappings between Hilbert spaces that have no points of Frechét differentiability. On the other hand, if we consider a map taking values in a metric space, the differential properties cannot be interpreted in classical terms.

We start by recalling in Section VI.1 the concept of Gaussian measure. After that, we will give in Section VI.2 some basic definitions related to the Wiener space structure. We then define in Section VI.3 $\mathcal{H}$-Lipschitzian maps and compare it with some Sobolev classes over Gaussian measures. In Section VI.4 we recall the definition of metric differentiability and w^*-differentiability and we finish in Section VI.5 by giving a Rademacher theorem in this context.

VI.1 Gaussian measures

The most natural measure in a finite-dimensional linear space is the Lebesgue measure. However, it is not a probability measure (the measure of the whole space is infinity, not 1), and it fails to exist in infinite dimension. Indeed, in an infinite-dimensional separable Banach space, every translation-invariant measure that is not identically zero has the property that all open sets have infinite measure. To see this, suppose that for some ε, the open ball of radius ε has finite measure. Since the space is infinite dimensional, one can construct an infinite sequence of disjoint open balls of radius $\varepsilon/4$ which are contained in the ε-ball (we can use the same argument that the one used to prove that the unit ball in an infinite dimensional space is not compact). Since we suppose that each of these balls has the same measure, and the sum of their measures is finite, the $\varepsilon/4$-balls must have measure 0. Since the space is separable, it is Lindelöf, and so it can be covered with a countable collection of $\varepsilon/4$-balls. Thus the whole space must have measure 0.

In the absence of a reasonable translation-invariant measure on a given function space, one might hope that there is a measure which at least satisfies the following condition: every translate of a zero measure set also has measure zero. One kind of measures satisfying such condition are the so-called *Gaussian measures.*

The modern theory of Gaussian measures lies at the intersection of the theory of random processes, functional analysis, and mathematical physics and is closely connected with diverse applications in quantum field theory, statistical physics, financial mathematics, and other areas of sciences. The study of Gaussian measures combines ideas and methods from probability theory, nonlinear analysis, geometry, linear operators, and topological vector spaces in a beautiful and nontrivial way. (Preface, p. xi.) V.I. Bogachev, "Gaussian measures", AMS 1998.

Definition VI.1.1. A Borel probability measure γ on $\mathbb{R}$ is called *Gaussian* if it is either the Dirac measure δ_a at a point a or has density

$$\rho(\cdot, \mu, \sigma^2) : t \to \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(t-\mu)^2}{2\sigma^2}}$$

with respect to the Lebesgue measure. The parameter μ is called the *mean* and σ^2 is commonly known as *variance.* The measure with density $\rho(\cdot, 0, 1)$ is called *standard.* A mean zero gaussian measure is called *centered.*

Let $(E, \|\cdot\|)$ be a separable Banach space endowed with a Gaussian measure γ. A *Gaussian measure* γ on E equipped with its Borel σ-algebra $\mathscr{B}$ is a probability measure on $(E, \mathscr{B})$ such that the law (push-forward measure) of each continuous linear functional on E is Gaussian, that is, $\gamma \circ (e^*)^{-1}$ is a Gaussian measure on $\mathbb{R}$ for each $e^* \in E^* \setminus \{0\}$, possibly a Dirac mass. If we assume, as we shall do, that γ is not supported in a proper subspace of E, then all such measures are Gaussian measures.

Remark VI.1.2. Let us observe that if γ is not supported in a proper subspace of E, then $\gamma \circ (e^*)^{-1}$ cannot be a Dirac mass. Indeed, if $\gamma \circ (e^*)^{-1}$ is a Dirac mass concentrated on 0, then $\gamma \circ (e^*)^{-1}(0) = 1$, contradicting the fact that $(e^*)^{-1}(0) \neq E$.

We shall also assume, for the sake of simplicity, that γ is centered, i.e. $\int_E x d\gamma(x) = 0$. Note that since E is separable and γ is a Borel probability measure on E, γ is Radon, that is, for every $\varepsilon > 0$ there is a compact set $K \subset E$ such that $\gamma(K) \geq 1 - \varepsilon$.

VI.2 Abstract Wiener spaces

In this section we introduce the concept of an abstract Wiener space introduced by L. Gross [51]. Our starting point is (E, γ), a separable Banach space E endowed with a

Gaussian measure γ. The *Cameron-Martin space* associated to (E,γ) can be defined, as a vector space, by

$$\mathcal{H} := \Big\{ \int_E x\phi(x)d\gamma(x) : \phi \in L^2(\gamma) \Big\},$$

where the integral above, well defined thanks to Fernique's exponential integrability theorem (see [81, 4.1]), has to be understood as a Bochner integral (see definition in VI.2.3). Indeed,

$$\Big\| \int_E x\phi(x)d\gamma(x) \Big\| \leq \int_E \|x\||\phi(x)|d\gamma(x) \overset{\text{(Hölder)}}{\leq} \|\phi\|_{L^2(\gamma)} \Big(\int_E \|x\|^2 d\gamma(x) \Big)^{\frac{1}{2}}$$
$$\overset{(*)}{=} \|\phi\|_{L^2(\gamma)} \sup_{\substack{\|e_n^*\|\leq 1 \\ n\geq 1}} \Big(\int_E |\langle e_n^*, x\rangle|^2 d\gamma(x) \Big)^{\frac{1}{2}} < \infty,$$

where the equality $(*)$ holds since the norm on E can be described as a supremum over a countable set $\{e_n^*\}_{n\geq 1}$ of elements of the unit ball of the dual space E^*. Now, by Fernique's theorem,

$$\int_E \exp(c\|x\|^2)d\gamma(x) < \infty \quad \text{if and only if} \quad 2c < \big(\sup_{\|e^*\|\leq 1} \|\langle e^*, x\rangle\|_{L^2(\gamma)} \big)^{-1}$$

and so, $\| \int_E x\phi(x)d\gamma\| \leq c\|\phi\|_{L^2(\gamma)}$ with $c = c(\gamma)$. The Cameron-Martin space $\mathcal{H} = \mathcal{H}(\gamma)$ is also called in the literature, the *reproducing kernel Hilbert space* .

We denote by $i : L^2(\gamma) \to \mathcal{H} \subset E$ the map $\phi \to \int_E x\phi(x)d\gamma(x)$, and by K the kernel of i. Let us observe that, since i is continuous, K is closed in $L^2(\gamma)$. In addition, i is surjective by definition and it is not injective unless $E = 0$. Hence, we can consider the following isomorphism

$$\begin{array}{cccc} i & : & L^2(\gamma)/\ker i & \longrightarrow & \mathcal{H} \\ & & \phi & \longrightarrow & [\phi] \end{array}$$

and so we can define the Cameron-Martin norm

$$\|i(\phi)\|_{\mathcal{H}} = \min_{\psi\in K} \|\phi - \psi\|_{L^2(\gamma)},$$

whose induced scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ satisfies

$$\langle i(\phi), i(\psi)\rangle_{\mathcal{H}} = \int_E \phi\psi d\gamma \quad \forall \phi \in L^2(\gamma), \forall \psi \in K^{\perp}.$$

Observe that $\mathcal{H}$ is a Hilbert space which is continuously injected in E, because the continuity of i gives $\|h\| \leq c\|h\|_{\mathcal{H}}$. Since γ is not supported in proper subspaces of E it follows that $\mathcal{H}$ is a dense subset of E. Indeed, if $\text{supp}(\gamma) \subsetneq E$, then $\mathcal{H} \subset \text{supp}(\gamma)$ (if there exists $h \in \mathcal{H} \setminus \text{supp}(\gamma)$ then $\gamma(\text{supp}(\gamma) + h) = 1$). Now, since $\mathcal{H} \subset \text{supp}(\gamma) \subsetneq E$ and $\text{supp}(\gamma)$ is closed, if $\overline{\mathcal{H}} = E$ then we cannot have $\text{supp}(\gamma) \subsetneq E$.

Since i is not injective in general, it is often more convenient to work with the map $j^* : E^* \to \mathcal{H}$, dual of the inclusion map $j : \mathcal{H} \to E$ (i.e. $j^*(e^*)$ is defined by $\langle j^*(e^*), h\rangle_{\mathcal{H}} = \langle e^*, h\rangle$):

$$E^* \xrightarrow{j^*} \mathcal{H}^* \sim \mathcal{H} \xrightarrow{j} E.$$

It is easy to check that the set $j^*(E^*)$ is dense in $\mathcal{H}$ for the norm $\|\cdot\|_{\mathcal{H}}$ and the fact that j is dense implies that j^* is injective.

The triple $(E, \mathcal{H}, \gamma)$ is called an *abstract Wiener space*. Recall that, more in general, every Radon Gaussian measure γ on a locally convex space E possesses the so called Cameron-Martin space.

It became clear in the $1970'$s from the early work of L. Gross [51] on analysis on Banach spaces with Gaussian measures, that in such analysis the differentiation should be restricted to differentiation in directions given by the Cameron-Martin space. The following nice theorem shows the great importance of the space $\mathcal{H}$.

Theorem VI.2.1. (Cameron-Martin Theorem) *Let $v \in E$ and let $T_v\gamma(B) = \gamma(B+v)$ be the shifted measure. Then $T_v\gamma \ll \gamma$ if and only if $v \in \mathcal{H}$.*

This is a basic tool in discussing absolute continuity and singularity of a Gaussian measure and its translates. The theorem states that every translate of a zero measure set also has measure zero only for translations by elements of $\mathcal{H}$. Moreover, if $h \notin \mathcal{H}$ then $T_v\gamma \perp \gamma$ (see Theorem [31, 2.8]). It can be proved that if E is infinite dimensional, then $\mathcal{H}$ is much smaller than E, although it may be dense in E. In fact, $\gamma(\mathcal{H}) = 0$ (see [16, 2.4.7]). Recall also that $E = \mathcal{H}$ if and only if E is finite dimensional.

Let us observe that the Cameron-Martin space depends on the measure. If $\mu \sim \nu$ then $\mathcal{H}(\mu) = \mathcal{H}(\nu)$.

For the sake of completeness, we now describe the classical Wiener space.

Example VI.2.2. (Classical Wiener space) Let

$$E = \{f \in \mathcal{C}([0,1] : f(0) = 0\} \text{ with } \|\cdot\|_E = \|\cdot\|_\infty$$

be the path space of one-dimensional Brownian motion. Let $(W_t)_{t\in[0,1]}$ be a stardand Brownian motion, or Wiener process, starting at the origin. Observe that for a Wiener process one can take the function

$$W_t(f) = f(t) \quad t \in [0,1] \quad f \in E.$$

Recall that a *Wiener process* is a continuous-time stochastic process characterized by three facts:

(1) $W_0 = 0$.

(2) The trajectories $t \to W_t(f)$ are continuous for a.e. f.

(3) W_t has independent increments $W_t - W_s$ with Gaussian distribution with mean 0 and variance $t - s$, for $0 \leq t \leq s$.

The *Wiener measure* is the probability law on the space of continuous functions E induced by the Wiener process $(W_t)_{t\in[0,1]}$ and

$$\mathcal{H} = \{f \in E \cap AC([0,1]) : \int_{\mathbb{R}} |f'(t)|^2 dt < \infty\},$$

is its Cameron-Martin space with the inner product inherited by $L^2[0,1]$, i.e. $\langle f, g\rangle_{\mathcal{H}} = \langle f', g'\rangle_{L^2}$. Recall that by $f \in AC[0,1]$ we mean that f is absolutely continuous, that is, $f'(t)$ exists a.e. in $[0,1]$ and $f(t) = f(0) + \int_0^t f'(s)ds$.

The triple $(E, \mathcal{H}, \gamma)$ is called the *classical Wiener space*.

(VI.2.3) Bochner Integral Here we briefly recall the notion of the Bochner integral; see for example [34]. This notion extends the definition of Lebesgue integral to functions which take values in a Banach space.

Let (E, Σ, γ) be a measurable space and let V be a Banach space. If $f = \sum_{i=1}^k a_i \mathcal{X}_{E_i} : E \to V$ is a simple function, where E_i are disjoint members of the σ-algebra Σ, then the *Bochner integral* is defined as

$$\int_E f(x)dx = \sum_{i=1}^k a_i \gamma(E_i).$$

A measurable function $f : E \longrightarrow V$ is *Bochner integrable* if there exists a sequence s_n of simple functions such that

$$\lim_{n\longrightarrow\infty} \int_E \|f - s_n\|_B \, d\gamma = 0,$$

where the integral on the left-hand side is the ordinary Lebesgue integral. Let us denote by $L^p(E, V)$ the Banach space of all γ-measurable mappings f such that

$$\|f\|_{L^p(X,V)} := \Big\{ \int \|f\|_E^p \, \gamma(dx) \Big\}^{1/p} < \infty.$$

The following properties of the Bochner integral are well known:

$$\Big\| \int_E f(x)dx \Big\| \leq \int_E \|f(x)\|dx,$$

and

$$\Big\langle v^*, \int_E f(x)dx \Big\rangle = \int_E \langle v^*, f(x)\rangle \quad \forall v^* \in V^*.$$

VI.3 $\mathcal{H}$-Lipschitzian functions

We start by giving the definition of $\mathcal{H}$-Lipschitzian functions, Lipschitz functions only in the directions of the Cameron-Martin space.

Definition VI.3.1. Let $(E, \mathcal{H}, \gamma)$ be an abstract Wiener space and let (Y, d_Y) be a metric space. A mapping $f : E \to Y$ is said to be *$\mathcal{H}$-Lipschitzian at x* with constant C if it is Borel measurable and

$$d_Y(f(x+h), f(x)) \leq C\|h\|_{\mathcal{H}}, \quad \forall h \in \mathcal{H}.$$

If, for some constant C, f is $\mathcal{H}$-Lipschitzian at x with constant C for γ-a.e. x, then we say that f is $\mathcal{H}$-Lipschitzian with constant C.

We state in the next theorem two properties of $\mathcal{H}$-Lipschitzian functions; the first one corresponds, in this context, to Rademacher's theorem.

Theorem VI.3.2. [40],[16, 5.11.8] *Let $f : E \to \mathbb{R}$ be $\mathcal{H}$-Lipschitzian. Then*

(1) *there exists a Borel γ-negligible set $N \subset E$ such that, for all $x \in E \setminus N$, the map $h \mapsto f(x+h)$ is Gâteaux differentiable at* 0*;*

(2) *there exists a modification $\tilde{f}$ of f in a γ-negligible set which is $\mathcal{H}$-Lipschitzian at all $x \in E$.*

Since all our results will be outside zero-measure sets, in the following we will always consider the modification of the function f which is Lipschitzian at every point. Moreover, if we have an $\mathcal{H}$-Lipschitzian function with values in a separable Banach space Y we can always find a modification as in the real-valued case (see [16, 4.5.4]).

We summarize here the various types of differentiability that can be consider in this context.

Definition VI.3.3. Let $(E, \mathcal{H}, \gamma)$ be an abstract Wiener space, let Y be a normed space and let $\mathcal{M}$ be a certain class of nonempty subsets of E. A mapping $f : E \to Y$ is said to be differentiable with respect to $\mathcal{M}$ if there exists a continuous linear mapping from E to Y, denoted by $Df(x)$, such that, for every fixed set $M \in \mathcal{M}$, one has uniformly in h from M:

$$\lim_{t \to 0} \frac{f(x+th) - f(x)}{t} = D_h f(x).$$

Taking for $\mathcal{M}$ the collection of all finite sets, we get the *Gâteaux differentiability.* If $\mathcal{M}$ is the class of all compact subsets, we get the *Hadamard differentiability.* Finally, if $\mathcal{M}$ consists on all bounded sets, then we get the *Fréchet differentiability.* It is clear that in finite dimensional spaces the Hadamard definition is equivalent to the Fréchet one and is stronger than Gâteaux differentiability. In addition, a useful fact is that Gâteaux and Hadamard differentiability coincide for locally Lipschitzian mappings.

In addition we can define, as it is natural in this context, the differentiability along $\mathcal{H}$ (in the corresponding sense) at the point x as the differentiability at $h = 0$ of the mapping $h \mapsto f(x+h)$ from $\mathcal{H}$ to Y in the corresponding sense.

Let E be a separable Banach space. Let Y be a separable Banach space with the Radon-Nikodym property and let γ be a Radon Gaussian measure. The main results in this context are:

$f : E \to Y$ is locally Lipschitz	$\Longrightarrow$ f is Gâteaux differentiable (and Hadamard differentiable) except for a gauss null set [16, 5.11.1] $\Downarrow$ f is $\mathcal{H}$-Fréchet differentiable
$f : E \to Y$ $\mathcal{H}$-Lipschitzian	$\Longrightarrow$ $\mathcal{H}$-Gâteaux differentiable γ-a.e.[16, 5.11.2] $\Longrightarrow$ $\mathcal{H}$-Hadamard differentiable γ-a.e. (Lemma VI.3.4) $\not\Rightarrow$ $\mathcal{H}$-Frechét differentiable γ-a.e.[16, 5.11.4]

Let us notice that all the statements remain valid if $\mathcal{H}$ is replaced by an arbitrary normed space F that is linearly embedded into E in such a way that F contains a countable everywhere dense set from $\mathcal{H}$.

Lemma VI.3.4. *If $f : E \to Y$ is $\mathcal{H}$-Lipschitzian then f is $\mathcal{H}$-Hadamard differentiable γ-a.e.*

Proof. Let K be a compact subset of $\mathcal{H}$. We have to prove that

$$\lim_{t\to 0} \left\| \frac{f(x+th) - f(x) - t \cdot D_h f(x)}{t} \right\| = 0$$

uniformly in $h \in K$ γ-a.e. x. It is enough to check that this is true for every sequence $\{t_n\} \to 0$. Let

$$f_n(h) = \frac{f(x+t_n h) - f(x) - t_n \cdot D_h f(x)}{t_n}.$$

Since f is $\mathcal{H}$-Lipschitzian, we have by Lemma [16, 4.5.2] that

$$\|f_n(h) - f_n(k)\| = \left\| \frac{f(x+t_n h) - f(x+t_n k)}{t_n} - D_{h-k} f(x) \right\| \le 2\,C \|h-k\|_{\mathcal{H}} \quad \forall h,k \in K.$$

Therefore, f_n is equicontinuous on K and it converges pointwise to 0. Thus, f_n converges uniformly on K to 0 (see p. 232 in [74]). $\square$

(VI.3.5) $\mathcal{H}$-Lipschitzian functions versus Sobolev spaces. Now, we mention the relation between real-valued $\mathcal{H}$-Lipschitzian functions and the Sobolev space $W^{1,\infty}_{\mathcal{H}}(E,\gamma)$ for Gaussian measures (see [16, Section 5.2]). In what follows, the weak $\mathcal{H}$-derivative for the Sobolev spaces will be denoted by $\nabla_{\mathcal{H}}$.

Theorem VI.3.6. *If $f \in W^{1,\infty}_{\mathcal{H}}(E,\gamma)$, then there exists a modification $\tilde{f}$ of f in a γ-negligible set which is $\mathcal{H}$-Lipschitzian at all $x \in E$, with constant $C = \text{ess-sup}\,|\nabla f|_{\mathcal{H}}$. Conversely, all $\mathcal{H}$-Lipschitzian functions $f : E \to \mathbb{R}$ belong to $W^{1,\infty}_{\mathcal{H}}(E,\gamma)$.*

We are going to study the differentiability properties of $\mathcal{H}$-Lipschitz functions $f : E \to Y$, where $(E,\mathcal{H},\gamma)$ is an abstract Wiener space and Y is a separable metric space or the dual of a separable Banach space. To this aim, following the same approach of [1] and [91], we introduce the Sobolev class $W^{1,\infty}_{\mathcal{H}}(E,\gamma,Y)$, where Y is any metric space, via the connection with the $\mathbb{R}$-valued Sobolev space $W^{1,\infty}_{\mathcal{H}}(E,\gamma)$.

Definition VI.3.7. Let $(E,\mathcal{H},\gamma)$ be an abstract Wiener space, let (Y,d_Y) be a metric space and let $\mathcal{F}$ be the collection of all 1-Lipschitz maps between Y and $\mathbb{R}$. Then, a Borel function $f : E \to Y$ belongs to $W^{1,\infty}_{\mathcal{H}}(E,\gamma,Y)$ if the following two condition hold:

(1) $\phi \circ f \in W^{1,\infty}_{\mathcal{H}}(E,\gamma)$ for each $\phi \in \mathcal{F}$.

(2) There exists $C \geq 0$ such that $\|\nabla_{\mathcal{H}}(\phi \circ f)\|_\infty \leq C$ for each $\phi \in \mathcal{F}$.

Recall that any separable metric space (Y,d_Y) embeds isometrically in duals of separable Banach spaces, for example in $\ell_\infty(\mathbb{N}) = (\ell_1(\mathbb{N}))^*$. A possible embedding is for instance given by the map

$$x \to \{d_Y(x,x_i) - d_Y(x_0,x_i)\}$$

where $\{x_i\}$ is a dense sequence in Y and $x_0 \in Y$ is a base point. The next result provides an extension of Theorem VI.3.6 when the target is the dual of a separable Banach space. By the above-mentioned isometric embedding theorem, the result applies also to maps with values in separable metric spaces.

Proposition VI.3.8. *Let $(E,\mathcal{H},\gamma)$ be an abstract Wiener space and let $Y = G^*$ be a dual Banach space, with G separable. If $f \in W^{1,\infty}_{\mathcal{H}}(E,\gamma,Y)$, then f has a Borel modification $\widetilde{f}$ in a γ-negligible set with*

$$\|\widetilde{f}(x+h) - \widetilde{f}(x)\|_Y \leq C\|h\|_{\mathcal{H}}, \quad \forall h \in \mathcal{H} \quad \forall x \in E.$$

Proof. Let $D \subset G$ be a dense and countable vector space over $\mathbb{Q}$. First, we define the function

$$\begin{array}{rccc} \varphi_g & : & Y & \longrightarrow & \mathbb{R} \\ & & x & \longrightarrow & \langle x, g\rangle, \end{array}$$

which is $\|g\|$-Lipschitz for any $g \in D$. Since $f \in W^{1,\infty}_{\mathcal{H}}(E,\gamma,Y)$, the function $f_g = \varphi_g \circ f \in W^{1,\infty}_{\mathcal{H}}(\gamma)$ for each $g \in D$. We know that $W^{1,\infty}_{\mathcal{H}}(E,\gamma)$ can be canonically identified with the space of $\mathcal{H}$-Lipschitzian functions. Moreover, by Theorem VI.3.2(2) we have that there exists a modification $\widetilde{f_g}$ of f_g which is $\mathcal{H}$-Lipschitz at each $x \in E$.

Let us denote

$$N_{g,g'} = \{x \in E : \widetilde{f_{g+g'}}(x) \neq \widetilde{f_g}(x) + \widetilde{f_{g'}}(x)\},$$

which is, thanks to the identity $f_{g+g'} = f_g + f_{g'}$, a γ-negligible set. Now, we are going to construct a full measure set $F_{g,g'} \subset E \setminus N_{g,g'}$ such that $F_{g,g'}$ is $\mathcal{H}$-invariant, that is, $F_{g,g'} + \mathcal{H} = F_{g,g'}$. Let us take an orthonormal basis $\{e_n\}$ in $\mathcal{H}$. Denote by $\{h_n\}$ the countable set of all finite linear combinations of the vectors e_i with rational coefficients. The set

$$\Omega_n = \{x \in E \setminus N_{g,g'} : x + h_n \in E \setminus N_{g,g'}\},$$

has full measure. If we put $F_{g,g'} = \bigcap_{n\in\mathbb{N}} \Omega_n$, then $F_{g,g'}$ has full measure as well and it is $\mathcal{H}$-invariant. Indeed, let $x \in F_{g,g'}$ and let $h \in \mathcal{H}$. We have to check that $x + h \in F_{g,g'}$. Let us choose a sequence $\{h_n\}$ converging to h in the norm of $\mathcal{H}$ such that

$$\widetilde{f_{g+g'}}(x+h_n) = \widetilde{f_g}(x+h_n) + \widetilde{f_{g'}}(x+h_n).$$

Now, since $\widetilde{f_{g+g'}}, \widetilde{f_g}, \widetilde{f'_g}$ are $\mathcal{H}$-Lipschitz functions we have that

$$\begin{aligned}
|\widetilde{f_{g+g'}}(x+h) - \widetilde{f_g}(x+h) - \widetilde{f_{g'}}(x+h)| =& |\widetilde{f_{g+g'}}(x+h) - \widetilde{f_g}(x+h) - \widetilde{f_{g'}}(x+h) \\
&+ \widetilde{f_{g+g'}}(x+h_n) - \widetilde{f_g}(x+h_n) - \widetilde{f_{g'}}(x+h_n)| \\
\leq& 3C\|h_n - h\|_{\mathcal{H}}
\end{aligned}$$

If we let n tend to infinity, we get that

$$\widetilde{f_{g+g'}}(x+h) = \widetilde{f_g}(x+h) + \widetilde{f_{g'}}(x+h),$$

as wanted. Observe that it $F_{g,g'}$ is $\mathcal{H}$-invariant, then $E \setminus F_{g,g'}$ is also $\mathcal{H}$-invariant. Since D is countable, the Borel set $N := \bigcup_{g,g'\in D}(E \setminus F_{g,g'})$ is γ-negligible and $\mathcal{H}$-invariant (since $X \setminus N = \bigcap_{g,g'} E \setminus F_{g,g'}$ is an intersection of $\mathcal{H}$-invariant sets).

Now, consider the functional

$$T : g \in D \longrightarrow \widetilde{f_g}(x),$$

which is $\mathbb{Q}$-linear in D for each $x \notin N$. In addition, we have that T is continuous. Indeed,

$$|\widetilde{f_g}(x)| \leq \sup_{x\in E} |\widetilde{f_g}(x)| = \|\widetilde{f_g}\|_\infty = \|f_g\|_\infty = \sup_{x\in E}\langle f(x), g\rangle = \langle \|f\|_\infty, g\rangle \leq C'\|g\|,$$

for each $g \in D$. Hence, it is the restriction to D of a linear continuous functional on G. Now, define $\widetilde{f}(x)$ as the unique element in G^* such that

$$\langle \widetilde{f}(x), g\rangle = \widetilde{f_g}(x) \quad \text{if } x \notin N$$

and $\widetilde{f}(x) = 0$ if $x \in N$. In order to prove that $\widetilde{f}$ is $\mathcal{H}$-Lipschitzian for each $x \in E$, just observe that

$$|\langle \widetilde{f}(x+h) - \widetilde{f}(x), g\rangle| = |\widetilde{f}_g(x+h) - \widetilde{f}_g(x)| \leq C\|h\|_{\mathcal{H}}\|g\| \quad \text{if } x \notin N.$$

Now, since by hypothesis the $\mathcal{H}$-Lipschitz constant is uniformly bounded for each $g \in G$, we have, upon taking the supremum over G, that $\widetilde{f}$ is an $\mathcal{H}$-Lipschitz function at each $x \in E$. □

VI.4 Metric differentiability and w^*-differentiability

Now we discuss the differentiability properties of $\mathcal{H}$-Lipschitzian maps. First of all notice that, if we consider a mapping taking values in a metric space admitting no linear structure, then the differential properties cannot be interpreted in classical terms. It turns out that, the local behavior of mappings of $\mathbb{R}^n$ into metric spaces can also be read in terms of the so called *metric differential* introduced in [77] (see also [78]).

Definition VI.4.1. Let $f : \mathbb{R}^k \to Y$, where (Y, d_Y) is any metric space. We shall denote

$$mdf_x(u) = \lim_{t\to 0} \frac{d_Y(f(x+tu), f(x))}{|t|} \tag{VI.1}$$

wherever this limit exists. We say that f is *metrically differentiable* at x if (VI.1) exists for all $u \in \mathbb{R}^k$ and $mdf_x(\cdot)$ is a continuous seminorm on $\mathbb{R}^k$.

Example VI.4.2. Let $Y = \mathbb{R}^n$ with the usual Euclidean distance and $k = 1$. If γ is differentiable at the point t, then the metric derivative at the point t is simply the Euclidean norm of the derivative of γ at the point t.

A Lipschitz function from an interval to a Banach space need not be differentiable somewhere. Indeed, if we consider $f : [0,1] \longrightarrow L^1[0,1]$, $t \mapsto \chi_{[0,t]}$, the difference quotients at $t \in \mathbb{R}$ do not converge in $L^1[0,1]$. However, the notion of metric differentiability allows to give a generalization of the classical Rademacher's theorem.

Theorem VI.4.3. [3, 3.2],[77],[78] *Any Lipschitz function $f : \mathbb{R}^k \to Y$ is metrically differentiable at $\mathscr{L}^k$-a.e. $x \in \mathbb{R}^k$.*

In [35] this theorem has been generalized to mappings between Banach spaces, when the domain is separable. Another different generalization has been given in [89], when the domain is a Carnot group. In that work, a metric differentiability theorem is obtained, as it is natural in that context, along the "horizontal" directions.

The property we look for is the natural transposition in our context of the one given in Definition VI.4.1:

Definition VI.4.4. Let $(E, \mathcal{H}, \gamma)$ be an abstract Wiener space and let (Y, d_Y) be a metric space. We say that $f : E \to Y$ is *metrically differentiable* at x if there exists a continuous seminorm $mdf_x(\cdot)$ in $\mathcal{H}$ such that

$$mdf_x(h) = \lim_{t\to 0} \frac{d_Y(f(x+th), f(x))}{|t|} \qquad \forall h \in \mathcal{H}.$$

As we have mentioned before, using an isometric embedding of Y in a dual space, we reduce ourselves to the case of duals of separable Banach spaces; the linear structure we gain allows us to give a metric differentiability theorem through a weaker version of differentiability for maps with values in dual Banach spaces, namely w^*-differentiability. It seems that this notion goes back to [61].

Definition VI.4.5. Let $(E, \mathcal{H}, \gamma)$ be an abstract Wiener space and let $Y = G^*$ be a dual Banach space. A function $f : E \to Y$ is *w^*-differentiable* at $x \in E$ if there exists a continuous linear map $wdf_x : \mathcal{H} \to Y$ satisfying

$$\frac{f(x+th) - f(x) - t \cdot wdf_x(h)}{t} \xrightarrow{w^*} 0 \ \text{ as } \ t \to 0, \quad \forall h \in \mathcal{H}.$$

Notice that if $f : [0,1] \mapsto L^1[0,1]$, $t \mapsto \chi_{[0,t]}$ is viewed as a map with values in the space $(C([0,1])^*$ of Radon measures in $[0,1]$, then f is metrically differentiable and w^*-differentiable with $wdf_x(t) = t\delta_x$.

Remark VI.4.6. Metric differentiability and w^*-differentiability have their own interest itself. In addition, if Y is uniformly convex, metric differentiability together with w^*-differentiability implies Frechét differentiability.

At a given point, the metric differential and the w^*-differential at a given point are related by

$$\|wdf_x(h)\| \le mdf_x(h) \qquad \forall h \in \mathcal{H}.$$

VI.5 Rademacher's theorem for $\mathcal{H}$-Lipschitzian functions

The following simple lemma will be useful in the sequel.

Lemma VI.5.1. *Let N be a Borel set in E and let $h \in j^*(E^*)$ be a vector with unit norm. If $\mathscr{L}^1(\{t \in \mathbb{R} : x + th \in N\}) = 0$ for each $x \in E$ then $\gamma(N) = 0$.*

(VI.5.2) Decomposition of a gaussian measure. Before proving the Lemma above, we recall a useful tool which allows us to decompose a measure into more elementary components. This process involves the notion of *conditional measure.* Let $h \in j^*(E^*) \subset \mathcal{H}$ be a vector with unit norm. We can define the following linear projection:

$$\begin{array}{rcccl} \pi & : & K \oplus \mathbb{R}h & \longrightarrow & \mathbb{R}h \\ & & x & \longrightarrow & \pi(x) = \langle e^*, x \rangle h \end{array}$$

where K is the kernel of e^*, and we can identify E with $K \oplus \mathbb{R}h$ (since $\langle e^*, h \rangle = \langle h, h \rangle_{\mathcal{H}} = 1$ we obtain that $\pi \circ \pi = \pi$). Now, we define the natural projection $\pi_K : E \to K$ by $x \mapsto x - \pi(x)$ and denote by ν the image of the measure γ under the projection π_K. In [16, 3.10.2] it is proved that the *conditional measures* γ^y ($y \in K$), characterized up to ν-negligible sets by the property of being concentrated on $y + \mathbb{R}h$ and by

$$\gamma(B) = \int_K \gamma^y(B) d\nu(y) \qquad \forall B \in \mathscr{B}(E),$$

can be explicitly represented by

$$\gamma^y(B) = \gamma_1(\{t \in \mathbb{R} : \ y + th \in B\})$$

where γ_1 denotes the standard Gaussian measure on $\mathbb{R}$.

Proof of Lemma VI.5.1. Using the disintegration of the measure γ described above we have that

$$\begin{aligned} \gamma(N) &= \int_K \gamma^y(N) d\nu(y) = \int_K \gamma_1(\{t \in \mathbb{R} : \ th + y \in N\}) d\nu(y) \\ &= \int_K 0 d\nu(y) = 0 \qquad (\gamma_1 \ll \mathscr{L}^1). \end{aligned}$$

□

Now, we are in a position to prove a metric differentiability theorem in the context of abstract Wiener spaces.

Theorem VI.5.3. *Let (E, H, γ) be an abstract Wiener space and let $Y = G^*$ be a dual Banach space, with G separable. Let $f : E \to Y$ be $\mathcal{H}$-Lipschitz. Then $f : E \to Y$ is w^*-differentiable and metrically differentiable and*

$$mdf_x(h) = \|wdf_x(h)\|_Y \quad \forall h \in \mathcal{H}$$

for γ-a.e. $x \in E$.

Proof. We denote by $\bar{N}$ a Borel γ-negligible set such that f is $\mathcal{H}$-Lipschitz, with constant C, at all $x \in E \setminus \bar{N}$.

Let $D \subset G$ be a dense and countable vector space over $\mathbb{Q}$. First, we define the function

$$\begin{array}{rccl} f_g & : & E & \longrightarrow & \mathbb{R} \\ & & x & \longrightarrow & \langle f(x), g\rangle, \end{array}$$

which is $\mathcal{H}$-Lipschitz for any $g \in D$. Indeed, for $x \in E \setminus \bar{N}$ we have

$$|\langle f(x+h) - f(x), g\rangle| \leq \|f(x+h) - f(x)\|_Y \|g\|_G \leq C\|h\|_{\mathcal{H}}\|g\|_G \qquad \forall h \in \mathcal{H}.$$

By Stroock and Enchev's Rademacher's (Theorem VI.3.2(1)), there exists a Borel γ-negligible set $N_g \supset \bar{N}$ such that f_g is $\mathcal{H}$-differentiable (i.e. Gateaux differentiable, along the directions in $\mathcal{H}$) at all $x \in E \setminus N_g$. Since D is countable, the Borel set $N := \bigcup_{g\in D} N_g$ is γ-negligible as well and f_g is $\mathcal{H}$-differentiable at any $x \in E \setminus N$ for any $g \in D$.

Now, fix $h \in \mathcal{H}$ and consider the functional

$$L_h : g \longrightarrow L_h(g) = \nabla_h f_g(x),$$

where $\nabla_h f_g(x)$ denotes the directional derivative of f_g in the direction of h at x, that is,

$$\nabla_h f_g(x) := \lim_{t\to 0} \frac{f_g(x+th) - f_g(x)}{t}.$$

The functional L_h is $\mathbb{Q}$-linear in D and since L_h is continuous (because, for each $g \in D$, $\|\nabla_h f_g(x)\|_{\mathcal{H}} \leq C\|h\|_{\mathcal{H}}\|g\|_G$) it is the restriction to D of a linear continuous functional on G, that we represent by a vector $\beta_h \in Y$, with $\|\beta_h\|_Y \leq C\|h\|_{\mathcal{H}}$. Once more, $h \mapsto \beta_h$ is additive and continuous, so it corresponds to a continuous linear functional $\nabla f(x) : \mathcal{H} \to Y$. Summing up, for $x \in E \setminus N$ we have a continuous linear functional $\nabla f(x) : \mathcal{H} \to Y$ satisfying

$$\nabla_h f_g(x) = \langle \nabla f(x)(h), g\rangle \qquad \forall h \in \mathcal{H},\ g \in D.$$

Using the definition of differentiability, we have that

$$\lim_{t\to 0} \left\langle \frac{f(x+th) - f(x) - t\cdot\nabla f(x)(h)}{t}, g\right\rangle = \lim_{t\to 0} \frac{f_g(x+th) - f_g(x) - t\cdot\nabla_h f_g(x)}{t} = 0,$$

for each $x \in E \setminus N$, $h \in \mathcal{H}$ and $g \in D$. Now, let $g \in G$ and $\{g_n\}$ a countable dense set in the unit ball of G such that $g_n \to g$. We have that

$$\begin{aligned} \lim_{t\to 0} \left\langle \frac{f(x+th) - f(x) - t\cdot\nabla f(x)(h)}{t}, g\right\rangle &= \lim_{t\to 0} \left\langle \frac{f(x+th) - f(x) - t\cdot\nabla f(x)(h)}{t}, g - g_n\right\rangle \\ &+ \lim_{t\to 0} \left\langle \frac{f(x+th) - f(x) - t\cdot\nabla f(x)(h)}{t}, g_n\right\rangle \to 0. \end{aligned}$$

Indeed, since $g_n \in D$ the second term of the sum goes to zero. On the other hand, the first term goes also to zero since the difference quotients are bounded. So, we obtain that the w^*-limit of the difference quotients is 0 for each $h \in \mathcal{H}$, and so f is w^*-differentiable at any $x \in E \setminus N$ and $wdf_x = \nabla f(x)$.

Now, we are going to prove that f is metrically differentiable and

$$mdf_x(h) = \|wdf_x(h)\|_Y \quad \forall h \in \mathcal{H}$$

for γ-a.e. $x \in E$. Since we have already proved that f is w^*-differentiable at any $x \in E \setminus N$, we have that

$$\lim_{t \to 0} \frac{f(x+th) - f(x) - t \cdot wdf_x(h)}{t} \xrightarrow{w^*} 0 \quad \forall x \in E \setminus N$$

and so

$$\lim_{t \to 0} \frac{f(x+th) - f(x)}{t} \xrightarrow{w^*} wdf_x(h) \quad \forall x \in E \setminus N.$$

As a supremum of w^*-continuous functions, every dual norm is a w^*-lower semicontinuous function. Using this fact we have that

$$\|wdf_x(h)\|_Y \leq \liminf_{t \to 0^+} \frac{\|f(x+th) - f(x)\|_Y}{t} \quad \forall h \in \mathcal{H}\ \forall x \in E \setminus N. \tag{VI.2}$$

Now, let D' be a countable dense set in the unit sphere of $\mathcal{H}$. Let us see that, given $h \in \mathcal{H}$, for γ-a.e.x it holds

$$\mathscr{L}^1(\{\tau \in \mathbb{R} :\ x + \tau h \in N\}) = 0.$$

Indeed,

$$\begin{aligned}
\int_E \mathscr{L}^1(\{\tau :\ x + \tau h \in N\}) d\gamma(x) &= \int_{\mathbb{R}} \gamma(\{x :\ x + \tau h \in N\}) d\tau && \text{(Fubini's Theorem)} \\
&= \int_{\mathbb{R}} T_{\tau h}(\gamma)(\{x :\ x \in N\}) d\tau \\
&= \int_{\mathbb{R}} 0 d\tau = 0 && (T_{\tau h}\gamma \ll \gamma).
\end{aligned}$$

Hence, if we set $N^h := \{x : \mathscr{L}^1(\{\tau :\ x + \tau h \in N\}) > 0\}$ and $N' := \bigcup_{h \in D'} N^h$, it is obvious that $\gamma(N') = 0$.

By the Fundamental Theorem of Calculus for Lipschitz functions we obtain that

$$f_g(x+th) - f_g(x) = \int_0^t \frac{d}{d\tau} f_g(x + \tau h) d\tau \overset{(*)}{=} \int_0^t \nabla_h f_g(x + \tau h) d\tau,$$

for any $t > 0$, $h \in D'$, $g \in D$ and $x \in E \setminus (N \cup N')$. Observe that the identity $(*)$ above makes sense because we have chosen x outside the set N', and so the integrands in the two integrals are equal $\mathscr{L}^1$-a.e. in $\mathbb{R}$ for each $h \in D'$ and each $g \in D$. Moreover, we have that

$$\lim_{\varrho \to 0} \frac{1}{\varrho} \int_0^{\varrho} \|\nabla f(x+\tau h)(h)\|_Y d\tau = \|\nabla f(x)(h)\|_Y \quad (**),$$

outside a γ-negligible set $N'' \subset E$ for every $h \in D'$ and $g \in D$. Indeed, if we denote

$$\mathcal{N}_h := \{x \in E \setminus N : \ (**) \text{ does not hold } \},$$

we know by the Lebesgue differentiation theorem that $\mathscr{L}^1(\{t : x + th \in \mathcal{N}_h\}) = 0$ for each $x \in E \setminus (N \cup N')$. Now, by Lemma VI.5.1 we obtain that $\gamma(\mathcal{N}_h) = 0$ and if we set $N'' := \bigcup_{h \in D'} \mathcal{N}_h$ the assertion follows.

We have that

$$\begin{aligned}|\langle f(x+th) - f(x), g\rangle| =& |f_g(x+th) - f_g(x)| = \Big| \int_0^t \nabla_h f_g(x+\tau h) d\tau \Big| \\ &\leq \int_0^t |\nabla_h f_g(x+\tau h)| d\tau = \int_0^t |\langle \nabla f(x+\tau h)(h), g\rangle| d\tau\end{aligned}$$

for any $t > 0$, $h \in D'$, $g \in D$ and $x \in E \setminus (N \cup N')$. By density, and taking the supremum over all $g \in G$ in the extreme parts of the previous inequality we obtain that

$$\|f(x+th) - f(x)\|_Y \leq \int_0^t \|\nabla f(x+\tau h)(h)\|_Y d\tau.$$

If $x \notin (N \cup N' \cup N'')$ and $h \in D'$ we can divide both sides by t and let t tend to zero to obtain

$$\limsup_{t \to 0^+} \frac{\|f(x+th) - f(x)\|_Y}{t} \leq \|\nabla f(x)(h)\|_Y = \|wdf_x(h)\|_Y \quad \forall h \in D'.$$

Again, by density of D' in the unit sphere and 1-homogeneity of directional derivatives, the inequality above holds for any $h \in \mathcal{H}$. This, combined with (VI.2), proves the metric differentiability of f at γ-a.e. x. $\square$

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